WHAT IS A TOPOLOGICAL STACK?

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1. INTRODUCTION

Stacks were introduced by Grothendieck to provide a general framework for studying local-global phenomena in mathematics. In this sense, the theory of stacks generalizes sheaf theory of Serre and Cartan. The early stages of the development of the theory can be traced in the Ph.D thesis of Grothendieck's student Giraud on non-abelian cohomology [Gi]. Not long after their introduction, stacks were used by Deligne and Mumford [DeMu] in a rather different context, namely, in the study of moduli of algebraic curves. In this work, among other fundamental contributions, Deligne and Mumford proved that the moduli problem in question gives rise to what is today called a *Deligne-Mumford stack*, and they used this fact to prove irreducibility properties of certain moduli spaces of curves. Later, M. Artin [Ar] generalized Deligne-Mumford's work by introducing Artin stacks, which have ever since proved to be a vital tool in algebraic geometry, especially in the study of moduli problems, and also in the study of quotient spaces.

Every scheme is a Deligne-Mumford stack, and every Deligne-Mumford stack is an Artin stack. There are also other variants of these notions which go by the generic name of *algebraic stacks*.

In a nutshell, algebraic stacks are a new breed of spaces for algebraic geometers to work with, providing them with greater flexibility for performing constructions hitherto impossible in the category of schemes, while being manageable enough to allow the entire machinery of scheme theory to be applicable to them.

After the great success of the theory of *algebraic stacks*, and their wide application (say, in arithmetic geometry, mathematical physics, stable homotopy theory,...), it was most natural to have parallel theories in other areas of mathematics in which there is a notion of *space* involved. To name a few, *analytic stacks* in the context of analytic geometry, *differentiable stacks* in the context of geometry of manifolds, *topological stacks* in the context of topology, and so on. Some of these theories have been (partly) developed.

Topological stacks, which had already been heuristically used (without even having been defined!) by some authors, were introduced in [No]. In *loc. cit.* it is shown that the usual homotopy theory of topological spaces can be extended to topological stacks. So, for instance, we can talk about homotopy groups of a topological stack.

Topological stacks provide a suitable framework for studying equivariant theories in topology. Also, they incorporate large classes of already well-known objects such as orbifolds [Th] and graph of groups [Se] (more generally, complexes of groups [Ha]), giving a unified perspective to these theories, as well as enabling one to apply homotopical methods in studying them. Also, every algebraic stack (defined over complex numbers) has an"underlying topological stack". This enables us to talk about topological invariants of algebraic stacks.

In this note we discuss certain features of the theory of topological stacks.

2. Topological stacks

We begin with the following not-so-illuminating definition. The reader unfamiliar with stack theory jargon may want to skip this definition and the paragraph that comes after it. A quick reading through would not be too harmful though!

Definition 2.1. A **topological stack** $\mathcal{X}$ is a stack on the site of topological spaces satisfying the following conditions:

1. The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

2. There exists a topological space X and an epimorphism $p\colon X \to \mathcal{X}$ which is representable by local Serre fibrations.

Let us say a few words about the above definition. By a *local Serre fibration* $f\colon X \to Y$ between topological spaces we mean a continuous map for which there exist coverings $X = \cup_{i \in I} U_i$ and $Y = \cup_{i \in I} V_i$ such that for each $i \in I$, $f|_{U_i}$ factors through V_i and induces a Serre fibration $f_i\colon U_i \to V_i$. This notion is local on the target and is invariant under base change, so it can be extended to representable morphisms of stacks. In particular, since in the presence of condition (**1**) the map p of (**2**) is representable, we can talk about p being a local Serre fibration. Therefore, the above definition makes sense!

Of course for a reader not previously exposed to the notion of a stack, the above definition and the discussion afterwards wouldn't make much sense. For this reason, for the rest of this exposition we will set aside the official definition of a topological stack and take a more informal approach.

What is essential to come to terms with is the fact that, with some extra care, one could do with topological stacks pretty much everything that can be done with usual topological spaces. For instance, we can talk about properties of morphisms of topological stacks such as being *open, closed, proper, an embedding, a Serre fibration, a covering space*, and so on. The point is that all these properties are invariant under base extension and are local on the target. So, we can define a representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of topological stacks to have property $\mathbf{P}$, if its base extension $f_Y\colon Y \times_{\mathcal{Y}} \mathcal{X} \to Y$ along any morphism $Y \to \mathcal{Y}$, with Y now being a topological space, has the property $\mathbf{P}$.

In particular, we can talk about substacks (respectively, open substacks, closed substacks) of a topological stack.

A topological space is naturally a topological stack. This can be expressed more precisely as a Yoneda type statement which asserts that the category of topological spaces can be identified with a full subcategory of the category of topological stacks. In particular, we can talk about homotopy groups of a topological stack. For instance, $\pi_1(\mathcal{X}, x)$ is defined to be the set $[(S^1, *), (\mathcal{X}, x)]_*$ of pointed homotopy classes of maps from the circle S^1 to $\mathcal{X}$. It can be shown that $\pi_1(\mathcal{X}, x)$ has a group structure and is functorial. (Of course, for this to make sense we need to have a notion of homotopy between maps of topological stacks. This can be done, but we will omit the definition here.)

A main class of examples of topological stacks are obtained as follows. Let G be a topological group acting continuously on a topological space X. To this action we associate what is called the **quotient stack** of this action, and is denoted by $[X/G]$. This should not be confused with the quotient space X/G we are familiar with from topology. The quotient stack $[X/G]$ is better behaved than X/G and retains much more information about the action than the ordinary quotient X/G. This is especially the case when the action has fixed points or misbehaving orbits. For instance, $[X/G]$ in some sense remembers all the stabilizer groups of the action, while X/G is completely blind to them. There is a natural morphism $\pi_{mod}\colon [X/G] \to X/G$ enabling us to compare the *stacky* quotient $[X/G]$ with the *coarse* quotient X/G. We will encounter π_{mod} again later in this note.

We can produce a whole lot of new topological stacks by gluing quotient stacks along their open substacks (the same way we produce manifolds by gluing copies of $\mathbb{R}^n$ along their open subspaces). Let us tentatively call such stacks *locally quotient stacks*. A special case of interest is when the group G is discrete and the action is properly discontinuous. Such locally quotient stacks are called **Deligne-Mumford** topological stacks and are of great importance. We say that a topological stack is **uniformizable** if it is of the form $[X/G]$ where G is a discrete group acting properly discontinuously on a topological space X. So every Deligne-Mumford topological stack is locally uniformizable. There are examples of Deligne-Mumford topological stacks that are not globally uniformizable.

It is an easy exercise to show that any orbifold or any graph of groups (or more generally, any complex of groups) gives rise to a Deligne-Mumford topological stack.

For the sake of simplicity, for the rest of this note we restrict ourselves to locally quotient stacks.

Let us try to give an intuitive picture of what goes into the structure of a topological stack. Every topological stack $\mathcal{X}$ has an underlying topological space, sometimes called the **coarse moduli space**, which we denote by X_{mod}. There is a natural functorial map $\pi_{mod}\colon \mathcal{X} \to X_{mod}$, called the *moduli map*. Loosely speaking, X_{mod} is the best approximation of $\mathcal{X}$ by a topological space. When $\mathcal{X} = [X/G]$ is a quotient stack, X_{mod} is simply the coarse quotient X/G.

A striking feature of topological stacks, which makes them adaptable to applications, and partly justifies their importance, is the following. Assume that $\mathcal{X} = [X/G]$ is a quotient stack. Then there is a natural quotient map $q\colon X \to [X/G]$, and this map, no mater how pathological the action is, makes X a principal G-bundle over $[X/G]$. So, in particular, $q\colon X \to [X/G]$ is a Serre fibration. The usual quotient map we know from topology is obtained as the composition $\pi_{mod} \circ q$. Note that the latter can be an extremely badly behaved map (think of the translation action of $\mathbb{Q}$ on $\mathbb{R}$).

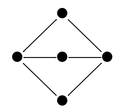
Now, consider a topological group G acting trivially on a point! The quotient stack $[*/G]$ of this action is called the **classifying stack** of G, and is denoted by BG. Observe that $(BG)_{mod}$ is just a point. However, BG is far from being a trivial object. More precisely, the quotient map $* \to BG$ makes $*$ into a principal G-bundle over BG, and this principal G-bundle is universal in the following sense: for every topological space T, the equivalence classes[1] of morphisms $T \to BG$ are in bijection with the isomorphism classes of principal G-bundles over T.

If in the above situation G is a discrete group, the quotient map $* \to BG$ becomes the universal cover of BG. This implies that $\pi_1 BG \cong G$.

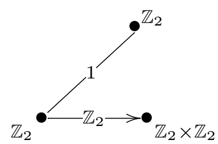
[1]Maps between two given stacks naturally form a groupoid. By an equivalence class we simply mean an isomorphism class in this groupoid.

Example 2.2. Consider the rotation action of the cyclic group $\mathbb{Z}_n$ on the sphere S^2 which fixes north and south poles. The quotient stack $\mathcal{X} = [S^2/\mathbb{Z}_n]$ has an underlying space which is homeomorphic to a sphere. However, $\mathcal{X}$ remembers the stabilizer groups at the two fixed points. Namely, at the north and south pole, $\mathcal{X}$ looks like $B\mathbb{Z}_n$. Outside these two *stacky* points, $\mathcal{X}$ is just like the sphere. We have $\pi_1\mathcal{X} \cong \mathbb{Z}_n$. The higher homotopy groups of $\mathcal{X}$ are isomorphic to those of S^2.

Example 2.3. Let X be the graph



and consider the action of the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on X in which the two generators act by flipping along the horizontal axis and 180 degree rotation, respectively. The quotient stack $\mathcal{X}$ of this action looks like this:



The decoration of the graph indicates that the quotient stack remembers the stabilizer group at each point. The coarse moduli stack of this action is the same graph (with the decoration deleted). The fact that X is a covering space of $\mathcal{X}$ gives us the following short exact sequence:

$$1 \to \mathbb{Z} * \mathbb{Z} \to \pi_1\mathcal{X} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1.$$

In fact, $\pi_1\mathcal{X}$ is isomorphic to the semi-direct product

$$(\mathbb{Z} * \mathbb{Z}) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

where the $(1,0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $\mathbb{Z} * \mathbb{Z}$ by switching the factors, and $(0,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ acts trivially. Alternatively, if we use van Kampen we obtain

$$\pi_1\mathcal{X} \cong \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

As an easy exercise in group theory, the reader can verify directly that this group is isomorphic to the previous one.

The higher homotopy groups of $\mathcal{X}$ are trivial.

The above examples suggest a rough way of visualizing a topological stack $\mathcal{X}$. Namely, we can think of $\mathcal{X}$ as a topological space (that would be X_{mod}) which at every point x is decorated with a (topological) group I_x. The group I_x is called the **stabilizer** (or **inertia**) group at x. These inertia groups are intertwined in an intricate way along X_{mod}. When $\mathcal{X}$ is a Deligne-Mumford topological stack, all I_x are discrete.

As we already pointed out, given a topological stack $\mathcal{X}$ and a point x on it, we can define a fundamental group $\pi_1(\mathcal{X}, x)$, which is functorial with respect to pointed maps. On the other hand, at every point x on $\mathcal{X}$, we have a pointed map $(BI_x, x) \to (\mathcal{X}, x)$. This induces a group homomorphism $\pi_1 \colon \pi_1(BI_x, x) \to \pi_1(\mathcal{X}, x)$. Assume now that $\mathcal{X}$ is Deligne-Mumford (hence I_x are discrete). Recall that, in this case we have $\pi_1(BI_x, x) \cong I_x$. So we get canonically-defined group homomorphisms $\omega_x \colon I_x \to \pi_1(\mathcal{X}, x)$.

We have the following theorem.

Theorem 2.4. *Let $\mathcal{X}$ be a Deligne-Mumford topological stack. Then $\mathcal{X}$ is uniformizable if and only if for every point x on $\mathcal{X}$ the homomorphism $\omega_x \colon I_x \to \pi_1(\mathcal{X}, x)$ is injective.*

The significance of the above result is that the maps ω_x are usually very easy to compute.

As a simple application of the above theorem let us see in an example how we can determine if a given orbifold is a *good orbifold* (in the sense of Thurston). That is, whether it is uniformizable.

Example 2.5. Fix a positive integer n. Let $\mathcal{X}$ be a stack whose underlying space is a torus and has a unique orbifold point of order n. Is this a good orbifold? To find the answer, first we compute the fundamental group of $\mathcal{X}$ using van Kampen. We obtain the following:

$$\pi_1\mathcal{X} \cong < a, b, c \mid aba^{-1}b^{-1} = c,\ c^n = 1 > .$$

There is only one orbifold point on $\mathcal{X}$, and the corresponding inertia group at this point is $\mathbb{Z}_n$. By Theorem 2.4, $\mathcal{X}$ is a good orbifold if and only if the map

$$\mathbb{Z}_n \to \pi_1\mathcal{X}$$
$$1 \mapsto c$$

is injective. Equivalently, we have to verify that the subgroup of $\pi_1\mathcal{X}$ generated by c is of order n. To do so, we show that there is a finite quotient H of $\pi_1\mathcal{X}$ such that the image of c in H has order n. We construct H as follows. Consider the action of $\mathbb{Z}_n$ on $\mathbb{Z}_n \oplus \mathbb{Z}_n$ where the action of the generator $1 \in \mathbb{Z}_n$ is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The semi-direct product $(\mathbb{Z}_n \oplus \mathbb{Z}_n) \rtimes \mathbb{Z}_n$, the Heisenberg group over $\mathbb{Z}_n$, is made into a quotient of $\pi_1\mathcal{X}$ by sending a,b and c to $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$, respectively. It is easy to check that this quotient has the desired property. In fact, using a more refined version of Theorem 2.4, this argument implies that there exists a compact surface X with a continuous action of H such that $\mathcal{X} \cong [X/H]$.

One can show that, in the above example $\mathcal{X}$ is the quotient of the upper half plane $\mathbb{H}$ by the action of a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

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