PICARD STACK OF A WEIGHTED PROJECTIVE STACK

BEHRANG NOOHI

Abstract. We prove that the Picard stack of a weighted projective stack over an arbitrary base scheme S is naturally isomorphic to $\mathbb{Z} \times B\mathbb{G}_{m,S}$

1. Introduction

The purpose of this note is to prove the following result.

Theorem 1.1. *Let S be an arbitrary base scheme. Then, there is a natural isomorphism of stacks over S*

$$\mathbb{Z} \times B\mathbb{G}_m \cong \mathcal{P}ic\big(\mathcal{P}_S(n_0, n_1, \cdots, n_r)\big),$$

where the left hand side means disjoint union of $\mathbb{Z}$ copies of the classifying stack of $\mathbb{G}_{m,S}$.

We present two proofs for this result. The first proof is completely elementary, but rather long. It occupies Section 2 to Section 5. The second proof (Section 6), which is essentially due to Angelo Vistoli, is a standard application of the (stack version of) Grothendieck's base change theorems.

Contents

2. Some elementary facts

In this section we prove a few lemmas which will be used in the course of the proof of the main theorem.

Lemma 2.1. *Let S be a scheme, and r a positive integer. Let $X = \mathbb{A}_S^{r+1}$ be the affine space over S, and $U = \mathbb{A}_S^{r+1} - \{0\}$ the complement of the zero section. Denote the base map $X \to S$ by p, the inclusion $U \hookrightarrow X$ by j, and the base map $U \to S$ by $q = p \circ j$.*

i. *If S is normal, then $p^* \colon \operatorname{Pic} S \to \operatorname{Pic} X$ is an isomorphism.*
ii. *If S is normal, then $q^* \colon \operatorname{Pic} S \to \operatorname{Pic} U$ is an isomorphism.*

iii. *Suppose* $S = \operatorname{Spec} K[\varepsilon]/(\varepsilon^2)$*, where* K *is a field. Then,* $\operatorname{Pic} X$ *is trivial. The same is true for* $\operatorname{Pic} U$ *if* $r \neq 1$*. (See Lemma 2.3 for the case* $r = 1$*.)*

Proof of (**i**). Let Cl stand for the divisor class group. We will make use of the fact that, for any integral scheme Y, the natural map $\mathrm{Pic}(Y) \to \mathrm{Cl}(Y)$ is injective. (This is an easy consequence of Krull Hauptidealsatz).

Let $\sigma \colon S \to X$ be a section for p, say given by $x_0 = \cdots = x_r = 1$. We claim that σ^* and p^* are inverse isomorphisms between $\mathrm{Pic}(X)$ and $\mathrm{Pic}(S)$. We know this is true in the case of class groups ([3], II. Proposition 6.6). In the case of Picard groups, the equality $\sigma^* \circ p^* = \mathrm{id}$ is obvious. The other equality follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "\sigma^*"] \arrow[d, hook] & \mathrm{Pic}(S) \arrow[r, "p^*"] \arrow[d, hook] & \mathrm{Pic}(X) \arrow[d, hook] \\
\mathrm{Cl}(X) \arrow[r, "\sigma^*"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Cl}(S) \arrow[r, "p_*"] & \mathrm{Cl}(X)
\end{tikzcd}$$

Proof of (**ii**). We show that the natural map $j^* \colon \mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism. When $r \geq 1$, this follows from ([3], II. Proposition 6.5.b). When $r = 0$, we know by ([3], II. Proposition 6.5.c) that j^* is surjective, and that its kernel is generated by the image of the divisor $Z : x_0 = 0$. But this divisor is trivial, so j^* is an isomorphism. To prove (**ii**) we can now simply repeat the argument of Part (**i**), with X replaced by U.

Proof of (**iii**). Let $X' = X_{red}$, that is $X' = \operatorname{Spec} K[x_1, \cdots, x_r]$. Similarly, set $U' = U_{red} = X' - \{0\}$. We will use the isomorphisms $H^1(X', \mathcal{O}_{X'}) \cong \operatorname{Pic} X$ and $H^1(U', \mathcal{O}_{U'}) \cong \operatorname{Pic} U$. For instance, in the latter case, this isomorphism can be realized as follows. Consider the standard covering $U = \cup_{i=0}^r U_i$, where $U_i = \operatorname{Spec} R[x_0, \cdots, x_r, x_i^{-1}]$. Let $\{f_i\}$ be a Čech 1-cocycle relative to this covering with values in $\mathcal{O}_{U'}$. Then, the 1-cocycle with values in $\mathcal{O}_U^*$ defined by $\{1 + f_i\varepsilon\}$ gives the corresponding line bundle on U.

Since X' is affine and $\mathcal{O}_{X'}$ is quasi-coherent, we have $H^1(X', \mathcal{O}_{X'}) = 0$, so $\operatorname{Pic} X$ is trivial. Same is true for U when $r = 0$, because in this case U is affine. We now compute $H^1(U', \mathcal{O}_{U'})$, for $r \geq 2$, using the Čech complex associated to the covering $U = \cup_{i=0}^r U_i$. This complex looks as follows:

$$\bigoplus_i K[x_0, \cdots, x_r, x_i^{-1}] \to \bigoplus_{i<j} K[x_0, \cdots, x_r, x_i^{-1}, x_j^{-1}] \to$$

$$\to \bigoplus_{i<j<k} K[x_0, \cdots, x_r, x_i^{-1}, x_j^{-1}, x_k^{-1}] \to \cdots$$

Writing out the cocycle condition for triple intersections $U_i \cap U_j \cap U_k$, and using the fact that $r \geq 2$, we see that a 1-cocyle in this complex is necessarily of the form $\{f_{i,j} - g_i + g_j\}_{i<j}$, where $f_{i,j} \in K[x_0, \cdots, x_r]$, and $g_i \in K[x_0, \cdots, x_r, x_i^{-1}]$ has the property that all of its monomials contain negative powers of x_i. This 1-cocycle is equal to the boundary of the 0-chain $\{h_i = f_{1,i} + g_i\}_i$, where $f_{1,1} := 0$. Thus, $\operatorname{Pic} U = H^1(U', \mathcal{O}_{U'})$ is trivial. □

Remark 2.2. Lemma 2.1.**i** and **ii** hold even if S is not normal. All we need is S be integral. The reason for assuming normal is to make life easier by quoting ([3], II. Proposition 6.6). The point is that, condition $(*)$ of *loc. cit.* is often redundant in the treatment of Weil divisors. For instance, for every $f \in R$, where R is a domain (that is not necessarily regular of codimension one), $\operatorname{div}(f)$ can be defined by

$$\operatorname{div}(f) = \sum_{\operatorname{ht} \mathfrak{p}=1} l_{\mathfrak{p}}(f) \cdot Y_{\mathfrak{p}},$$

where, by definition, $l_{\mathfrak{p}}(f)$ is the length of the $R_{\mathfrak{p}}$-module $R_{\mathfrak{p}}/(f)$, and $Y_{\mathfrak{p}}$ is the prime divisor corresponding to $\mathfrak{p}$. This definition has all the expected properties, and so does the corresponding notion of Weil divisor class group. With this definition of divisor class group, Proposition II.6.6 of [3], which is all we needed in the proof of Lemma 2.1, is true (with the same proof) for any integral scheme X.

Lemma 2.3. *Let K be a field, $R = K[\varepsilon]/(\varepsilon^2)$, and $S = \operatorname{Spec} R$. Let $U = \mathbb{A}^2_S - \{0\}$, and let $U = U_x \cup U_y$ be the standard covering of U. Namely, $U_x = \operatorname{Spec} K[x, y, x^{-1}]$ and $U_y = \operatorname{Spec} K[x, y, y^{-1}]$. Let V be the k-vector space spanned by the monomials $\{x^i y^j \mid i, j < 0\}$. For each $f \in V$, let L_f be the line bundle over U given by gluing the trivial line bundles over U_x and U_y along $U_x \cup U_y = \operatorname{Spec} K[x, y, x^{-1}, y^{-1}]$ by the function $1+\varepsilon f \in R[x, y, x^{-1}, y^{-1}]^{\times}$. Then, the correspondence $f \mapsto L_f$ induces an isomorphism $V \xrightarrow{\sim} \operatorname{Pic} U$.*

Proof. Use the Čech complex appearing in the proof of Lemma 2.1.**iii**. $\square$

Exercise. Notation being as in Lemma 2.3, let $s, t \in K$ be constants, with $t \neq 0$, and let m, n be arbitrary positive integers. Consider the automorphism $\varphi \colon U \to U$

$$x \to (t+\varepsilon s)^m x, \quad y \mapsto (t+\varepsilon s)^n y.$$

Prove that, for every $f \in V$, we have $\varphi^*(L_f) = L_{\varphi^* f}$, where $\varphi^* f$ is defined by $\varphi^* f(x, y) := f(t^m x, t^n y)$.

3. Line bundles on weighted projective stacks

Let $S = \operatorname{Spec} R$ be an affine scheme, and let $\mathbb{G}_m = \mathbb{G}_{m,S} = \operatorname{Spec} R[t, t^{-1}]$ be the multiplicative group scheme over S. Let $U := \mathbb{A}^{r+1} - \{0\}$, $r \geq 1$. Given a sequence of positive integers $n_0, n_1, \cdots, n_r$ we define the weight $(n_0, n_1, \cdots, n_r)$ action $\mathbb{G}_m \times U \to U$ of $\mathbb{G}_m$ on U by the ring homomorphism

$$R[x_0, \cdots, x_r] \to R[t, t^{-1}, x_0, \cdots, x_r],$$
$$x_i \mapsto t^{m_i} x_i.$$

In other words, t acts by multiplication by $(t^{n_0}, t^{n_1}, \cdots, t^{n_r})$.

The quotient stack of this action is called the *weighted projective stack* over S of weight $(n_0, n_1, \cdots, n_r)$, and is denoted by $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ or $\mathcal{P}_R(n_0, n_1, \cdots, n_r)$. Since this construction is local on S, we can talk about weighted projective stacks over an arbitrary base scheme S. (Equivalently, we can define $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ by pulling back $\mathcal{P}_{\mathbb{Z}}(n_0, n_1, \cdots, n_r)$ along $S \to \operatorname{Spec} \mathbb{Z}$.)

To give a line bundle on $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ is the same as to give a $\mathbb{G}_m$-equivariant line bundle on U. Recall that, a $\mathbb{G}_m$-equivariant line bundle on U means a line bundle L on U, together with an isomorphism $\varphi \colon pr^* L \to \mu^* L$, where $pr, \mu \colon \mathbb{G}_m \times U \to U$ are the projection and the multiplication morphisms, respectively. We require that φ satisfies the obvious cocycle condition on $\mathbb{G}_m \times \mathbb{G}_m \times U$.

Suppose that $S = \operatorname{Spec} R$, where R is a normal domain, or $R = K[\varepsilon]/(\varepsilon^2)$. By Lemma 2.1 every line bundle over U is trivial . Therefore, in this case, to give a line bundle over $\mathcal{P}_R(n_0, n_1, \cdots, n_r)$ is equivalent to giving an invertible element $f(t, x_0, \cdots, x_r) \in R[t, t^{-1}, x_0, \cdots, x_r]^\times$ which satisfies the following cocycle condition:

$$f(ts, x_0, \cdots, x_r) = f(s, t^{m_0}x_0, \cdots, t^{m_r}x_r)f(t, x_0, \cdots, x_r).$$

Such an f gives rise to the trivial line bundle if it is a coboundary. Namely, if there exists $h \in R[x_0, \cdots, x_r]^\times$ such that

$$f(t, x_0, \cdots, x_r) = h(t^{m_0}x_0, \cdots, t^{m_r}x_r)h(x_0, \cdots, x_r)^{-1}.$$

(Note that, when R is an integral domain, this means $h = a$ for some unit $a \in R^\times$. In particular, if two cocycle differ by a coboundary, they should actually be equal.)

In fact, we have been slightly sloppy in the above discussion, because, a priori, f is a function on $\mathbb{A}_R^{r+1} - \{0\}$ and it is not guaranteed to be a polynomial. But the following Hartogs like lemma validates our argument.

Lemma 3.1. *Let R be an arbitrary ring and f a global section of the structure sheaf of $U = \mathbb{A}_R^{r+1} - \{0\}$. If $r \geq 1$, then f extends uniquely to a global section of $\mathbb{A}_R^{r+1}$.*

Proof. Let $U_i = \operatorname{Spec} R[x_0, \cdots, x_r, x_i^{-1}]$ and consider the covering $U = \cup_{i=1}^n U_i$. We show that the restrictions $f_i := f|_{U_i}$ are polynomials for every i. To see this, observe that, except possibly for x_i, all variables occur with positive powers in f_i. To show that x_i also occurs with a positive power, pick some $j \neq i$ and use the fact that x_i occurs with a positive power in $f_j|_{U_i \cap U_j} = f_i|_{U_i \cap U_j}$.

So, all f_i actually lie in $R[x_0, \cdots, x_r, x_i^{-1}]$. Since $f_j|_{U_i} = f_i|_{U_j}$, it is obvious that all f_i are actually equal to each other and provide the desired extension of f to $\mathbb{A}_R^{r+1}$. $\square$

For any fixed integer $d \in \mathbb{Z}$, the polynomial

$$f_d(t, x_1, \cdots, x_r) = t^d$$

satisfies the above cocycle condition. The corresponding line bundle is denoted by $\mathcal{O}(d)$. The map $d \mapsto \mathcal{O}(d)$ gives rise a group homomorphism

$$\mathbb{Z} \to \operatorname{Pic}\left(\mathcal{P}_R(n_0, n_1, \cdots, n_r)\right).$$

In the case where R is a field, it is a well-known fact that this is an isomorphism. It follows from the main theorem of this paper that this is indeed true for an arbitrary local ring R. In the next lemma we prove a special case where R is normal.

Lemma 3.2. *Assume R is a normal local domain. Then any line bundle $\mathcal{L}$ over $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.*

Proof. Since every line bundle on $\mathbb{A}_R^{r+1} - \{0\}$ is trivial (Lemma 2.1.**ii**), $\mathcal{L}$ can be described by a cocycle $f(t, x_0, \cdots, x_r) \in R[t, t^{-1}, x_0, \cdots, x_r]$. That is,

$$f(ts, x_0, \cdots, x_r) = f(s, t^{m_0}x_0, \cdots, t^{m_r}x_r)f(t, x_0, \cdots, x_r).$$

We can write $f = t^d g$, where $d \in \mathbb{Z}$ and $g \in R[t, x_0, \cdots, x_r]$ is a polynomial that is not divisible by t. Since g also satisfies the cocycle condition, by substituting $s = 0$ in the cocycle condition and observing that the left hand side does not involve t, we see that g must be a constant polynomial. Of course, by the cocycle condition, the only choice for this constant is 1. Therefore, $f(t, x_1, \cdots, x_r) = t^d$. $\square$

4. Statement of the main result

For an arbitrary algebraic stack $\mathcal{X}$ over S, the Picard stack $\mathcal{P}ic(\mathcal{X})$ is the stack defined by associating to any $T \to S$ the groupoid of line bundles (and isomorphisms between them) on $\mathcal{X}_T = T \times_S X$. By decent theory, this is always a stack over S. When $\mathcal{X}$ is nice enough, it is indeed an algebraic stack.

Theorem 4.1 ([1], Theorem 5.1)**.** *Assume S is an affine scheme over an excellent Dedekind domain. If $\mathcal{X}$ is proper and flat over S, then $\mathcal{P}ic(\mathcal{X})$ is an algebraic stack.*

Picard stack is functorial, in the sense that $\mathcal{P}ic(\mathcal{X}_T) = T \times \mathcal{P}ic(X)$. Our goal is to study $\mathcal{P}ic\big(\mathcal{P}_S(n_0, n_1, \cdots, n_r)\big)$. We prove the following theorem

Theorem 4.2. *Let S be an arbitrary base scheme. Then, there is a natural isomorphism of stacks over S*

$$\mathbb{Z} \times B\mathbb{G}_{m,S} \cong \mathcal{P}ic\big(\mathcal{P}_S(n_0, n_1, \cdots, n_r)\big),$$

where the left hand side means disjoint union of $\mathbb{Z}$ copies of the classifying stack of $\mathbb{G}_{m,S}$.

Corollary 4.3. *A line bundle $\mathcal{L}$ on $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ is isomorphic to a line bundle of the form $f^*(\mathcal{M}) \otimes \mathcal{O}(d)$ for some $d \in \mathbb{Z}$, and a line bundle $\mathcal{M}$ on S, where $f\colon \mathcal{P}_S(n_0, n_1, \cdots, n_r) \to S$ is the projection map. Furthermore, d and the isomorphism class of $\mathcal{M}$ are uniquely determined by $\mathcal{L}$.*

Corollary 4.4. *Let $S = \operatorname{Spec} R$, where R is an arbitrary local ring. Then any line bundle on $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.*

5. Proof of the main theorem

Fix a base scheme S. Throughout this section we denote $\mathcal{P}ic\big(\mathcal{P}_S(n_0, n_1, \cdots, n_r)\big)$ by $\mathcal{P}$. Observe that, since $\mathcal{P}ic\big(\mathcal{P}_S(n_0, n_1, \cdots, n_r)\big) \cong S \times \mathcal{P}ic\big(\mathcal{P}_{\mathbb{Z}}(n_0, n_1, \cdots, n_r)\big)$, this is representable by an algebraic stack (Theorem 4.1).

Lemma 5.1. *Assume S is connected. Let $\mathcal{L}$ be a line bundle on $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$. Then, for any point $y\colon \operatorname{Spec} k \to S$, the restriction $\mathcal{L}_k$ of $\mathcal{L}$ to $\mathcal{P}_k(n_0, n_1, \cdots, n_r)$ is isomorphic to $\mathcal{O}(d)$, for some integer d that is independent of y.*

Proof. We may assume $S = \operatorname{Spec} R$ is affine. Also, since every line bundle is defined by finite data, we can find a finitely generated subring of R over which $\mathcal{L}$ is defined. So we may assume R is Noetherian. By replacing S with S_{red} we may assume S is reduced. Since it is enough to prove the statement for each irreducible component of S we may also assume that S is integral. Since the normalization of S surjects onto S we may assume S is normal. Finally, it is enough to prove the result for every local ring $\mathcal{O}_x$, $x \in S$. So we may further assume that R is local. The result now follows from Lemma 3.2. □

The above lemma shows that we have a decomposition $\mathcal{P} = \coprod_{d \in \mathbb{Z}} \mathcal{P}_d$. Let P be the fppf sheaf on $\mathbf{Sch}_S$ associated to the presheaf

$$V \mapsto \{\text{isomorphism classes in } \ \mathcal{P}(V)\}.$$

This is what is called the 'S-espace grossier' associated to $\mathcal{P}$ in [5]. Then, we have a decomposition $P = \coprod_{d \in \mathbb{Z}} P_d$, where each P_d is a subsheaf of P.

For any ring R, the global sections of $\mathcal{P}_R(n_0, n_1, \cdots, n_r)$ are precisely R. This is easy to show, because such global sections are precisely the global sections of

$\mathbb{A}^r_R$ which are $\mathbb{G}_m$-invariant. Therefore, we see that, for any integer d, the group of automorphisms of $\mathcal{O}(d)$ on $\mathcal{P}_R(n_0, n_1, \cdots, n_r)$ is $R^\times$. This implies that $\mathcal{P}$ is a $\mathbb{G}_m$-gerbe over P, and for each $d \in \mathbb{Z}$, $\mathcal{P}_d$ is a $\mathbb{G}_m$-gerbe over P_d.

Proposition 5.2. *For every $d \in \mathbb{Z}$ the natural map $P_d \to S$ is an isomorphism and $\mathcal{P}_d$ is a neutral $\mathbb{G}_m$-gerbe over P_d.*

Before giving the proof of the above proposition, let us note that the main result of the paper is an easy consequence

Theorem 5.3. *Let S be an arbitrary base scheme. The natural morphism*

$$\Psi \colon \mathbb{Z} \times B\mathbb{G}_m \xrightarrow{\sim} \mathcal{P}ic\big(\mathcal{P}_S(n_0, n_1, \cdots, n_r)\big)$$

which on the d-the component is given by the line bundle $\mathcal{O}(d)$ is an isomorphism of stacks over S.

The morphisms Ψ in the above theorem can be interpreted as follows. For any scheme T over S, a T-point of the left hand side corresponds, by definition, to a pair $(d, \mathcal{M})$, where d in an integer and $\mathcal{M}$ is a line bundle over T. Then, Ψ sends this to the T-point of $\mathcal{P}ic\big(\mathcal{P}_T(n_0, n_1, \cdots, n_r)\big)$ corresponding to the line bundle $f^*(\mathcal{M}) \otimes \mathcal{O}(d)$ on $\mathcal{P}_T(n_0, n_1, \cdots, n_r)$. Using this interpretation Corollary 4.3 follows immediately.

To prove Proposition 5.2 we need a couple of lemmas.

Lemma 5.4. *For every $d \in \mathbb{Z}$, the sheaf P_d is representable by an algebraic space and $f \colon P_d \to S$ is separated and locally of finite presentation.*

Proof. Representability by an algebraic space follows from Corollaire 10.8 of [5] because for every scheme U, the group scheme $\mathbb{G}_{m,U}$ is flat and of finite presentation over U. Separatedness of f follows from valuative criterion for separatedness, because f f induces a bijection on T-valued points whenever T is normal (use Lemma 3.2).

So all we need to show is that P_d is of locally of finite presentations. That is, we have to show that, for very projective system $\{U_i\}_{i \in I}$ of affine schemes (over S), the natural map

$$\varinjlim P_d(U_i) = P_d(\varprojlim U_i)$$

is a bijection. This follows from the fact that, for any affine U, a line bundle over $\mathcal{P}_U(n_0, n_1, \cdots, n_r)$ can be given by a 1-cocyle relative to the standard covering of $\mathcal{P}_U(n_0, n_1, \cdots, n_r)$ by affine patches (and so there are only finitely many data involved). ☐

Lemma 5.5. *Let $f \colon X \to S$ be a morphism of scheme. Assume f is locally of finite type, is bijective, and admits a section. Also, assume the fibers of f are reduced. Then f is an isomorphism. The same statement is true if X is an algebraic space and f is separated.*

Proof. We may assume S is a scheme. Let $x \in X$ be an arbitrary point. We prove that the induced map of local rings $f_x^* \colon \mathcal{O}_{f(x)} \to \mathcal{O}_x$ is an isomorphism. Let $A = \mathcal{O}_{f(x)}$ and $B = \mathcal{O}_x$, and let $m \subset A$ be the maximal ideal of A. Then, B is an A algebra and, by assumption, the natural map $A \to B$ admits a right inverse $g \colon B \to A$. Let $I := \ker g$ be the corresponding ideal in B. The assumption that f is bijective, implies that $A \to B$ induces a bijection on the set of prime ideals (and the inverse is provided by g). Therefore, I is a nilpotent ideal. On the other hand, we have a natural decomposition $B = A \oplus I$ as A-modules. The fact that f has reduced fibers implies that $M/mB = R/m \oplus I/mI$ is reduced. Therefore, $I/mI = 0$. So, if we prove that I is finitely generated, it follows from Nakayama's lemma that $I = 0$, which is what we want to prove.

To prove that I is finitely generated, note that B is of finite type over A. Let $\{a_i + t_i\}$ be a finite generating set, where $a_i \in A$ and $t_i \in I$. Let T be the set of all monomials in $\{t_i\}$. Then T generates I as an A-module. Finally, observe that T is finite because I is nilpotent.

To prove the second statement, note that f is quasi-finite, locally of finite presentation and separated. Therefore, by ([4], II. Corollary 6.16), X is a scheme. $\square$

Proof of Proposition 5.2. We show that the map $f\colon P_d \to S$ satisfies the hypotheses of Lemma 5.5.

We know by Lemma 5.4 that $P_d \to S$ is separated and locally of finite. The fact that f is a bijection follows from the fact that, for any field K, all line bundles over $\mathcal{P}_K(n_0, n_1, \cdots, n_r)$ are of the form $\mathcal{O}(d)$; see Lemma 3.2. A section for f is provided by the line bundle $\mathcal{O}(d)$ on $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$.

All that is left to check is that f has reduced fibers. Let K be a field and set $R = K[\varepsilon]/(\varepsilon^2)$. We must show that the only line bundle on $\mathcal{P}_R(n_0, n_1, \cdots, n_r)$ whose restriction to the special fiber is isomorphic to $\mathcal{O}(d)$ is itself isomorphic to $\mathcal{O}(d)$. To prove this, first we twist by $\mathcal{O}(-d)$ and reduce to the case $d = 0$. Let $\mathcal{L}$ be a line bundle on $\mathcal{P}_R(n_0, n_1, \cdots, n_r)$ whose restriction to the special fiber is $\mathcal{O}$. Let $f \in R[t, t^{-1}, x_0, \cdots, x_r]^\times$ be the cocycle representing $\mathcal{L}$, as in Section 3. (Here we are implicitly using Lemma 2.1.**iii** and Lemma 3.1; also see the exercise at the end of Section 2.) Recall that the cocycle condition means the following identity:

$$f(ts, x_0, \cdots, x_r) = f(s, t^{n_0}x_0, \cdots, t^{n_r}x_r)f(t, x_0, \cdots, x_r).$$

Since the restriction of $\mathcal{L}$ to the special fiber is trivial, we can, after modifying f, assume that $f = 1 + \varepsilon g$, where $g \in K[t, t^{-1}, x_0, \cdots, x_r]$ satisfies the following cocycle condition:

$$g(ts, x_0, \cdots, x_r) = g(s, t^{n_0}x_0, \cdots, t^{n_r}x_r) + g(t, x_0, \cdots, x_r).$$

We will show that this cocycle is a coboundary, in the sense that, there exists $h \in K[x_1, \cdots, x_r]$ such that

$$g(t, x_0, \cdots, x_r) = h(t^{m_0}x_0, \cdots, t^{m_r}x_r) - h(x_0, \cdots, x_r).$$

We can express g uniquely in the form

$$g(t, x_0, \cdots, x_r) = \sum_J g_J(t)x^J.$$

Here J runs through a finite set of r-tuples $(j_0, \cdots, j_r)$ of positive integers, x^J stands for the monomial $x_0^{j_0} \cdots x_r^{j_r}$, and $g_J(t)$ is a Laurant polynomial in t. The cocycle condition for g now reads

$$\forall J, \quad g_J(ts) = g_J(s)t^{J\cdot\mathbf{n}} + g_J(t),$$

where $\mathbf{n} = (n_0, \cdots, n_r)$, and $\cdot$ stands for dot product of vectors. The above identity implies that g_J is indeed a polynomial in t, because a negative power of t supplied by $g_J(t)$ on the right hand side can not correspond to anything on the left hand side. Set $a_J := -g_J(0) \in K$. By plugging $s = 0$ in the above equality, we see that $g_J(t) = a_J(t^{J\cdot\mathbf{n}} - 1)$. Set

$$h(x_0, \cdots, x_r) := \sum_J a_J x^J.$$

It is easily verified that

$$g(t, x_0, \cdots, x_r) = h(t^{m_0}x_0, \cdots, t^{m_r}x_r) - h(x_0, \cdots, x_r).$$

This proves that g is a coboundary. The proof is complete. □

Remark 5.6. The last part of the above proof can be interpreted as saying that $H^1\big(\mathcal{P}_R(n_0, n_1, \cdots, n_r), \mathcal{O}\big) = 0$, where $R = K[\varepsilon]/(\varepsilon^2)$.

6. Alternative approach

It was pointed out to me by Angelo Vistoli [7] that there is an easy way to prove Theorem 4.2 using Grothendieck's base change results ([3], III. Theorem 12.11). Here is how the proof goes in the case of the usual projective space $\mathbb{P}^r_S$. The appropriate modifications that are needed to make this proof applicable to the general case of weighted projective stacks will be explained afterwards.

An alternative proof of Theorem 4.2 for $\mathbb{P}^r_S$. We show that, for any connected base scheme S, a line bundle $\mathcal{L}$ on $\mathbb{P}^r_S$ is uniquely expressed in the form $f^*(\mathcal{M}) \otimes \mathcal{O}(d)$, where $\mathcal{M}$ is a line bundle on S and $f\colon \mathbb{P}^r_S \to S$ is the base map. Furthermore, under this identification, an isomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ is of the form $p^*(\psi) \otimes \mathrm{id}$, for a unique isomorphism $\psi\colon \mathcal{M} \to \mathcal{M}'$.

The proof of the above statement is quite easy using ([3], III. Theorem 12.11). First, we can reduce to the case where S is affine. Since a line bundle on $\mathbb{P}^r_S$ is given by finitely many data, we may assume S is finitely generated over $\mathbb{Z}$, and in particular Noetherian.

Step 1. First we prove that there exists an integer d such that $\mathcal{L}_x \cong \mathcal{O}(d)$ for every point $x \in S$. This of course was already proved in Lemma 5.1, but we give a direct proof.

Since S is connected, it is enough to prove that, for every $x \in S$, the statement is true in some neighborhood of x. After tensoring $\mathcal{L}$ by $\mathcal{O}(-d)$, we may assume that $\mathcal{L}_x \cong \mathcal{O}$. In particular, $H^1(\mathbb{P}^r_x, \mathcal{L}_x) = 0$. By semicontinuity, the same is true in some neighborhood of x. So, after shrinking to this neighborhood, we may assume it is true for every point on S. The maps

$$\phi^1(y)\colon R^1 f_*(\mathcal{L}) \otimes k(y) \to H^1(\mathbb{P}^r_y, \mathcal{L}_y)$$

are now surjective, for all $y \in S$. Hence, by ([3], III. Theorem 12.11.a), $\phi^1(y)$ must be an isomorphism for every $y \in S$. In other words, the fibers of the coherent sheaf $R^1 f_*(\mathcal{L})$ are zero at every point $y \in S$. This implies that $R^1 f_*(\mathcal{L}) = 0$. In particular, $R^1 f_*(\mathcal{L})$ is locally free. It follows now from ([3], III. Theorem 12.11.b, with $i = 1$), that, for every $y \in S$,

$$\phi^0(y)\colon f_*(\mathcal{L}) \otimes k(y) \to H^0(\mathbb{P}^r_y, \mathcal{L}_y)$$

is surjective, hence an isomorphism, by ([3], III. Theorem 12.11.a). On the other hand, $f_*(\mathcal{L})$ is locally free of rank 1, by ([3], III. Theorem 12.11.b, with $i = 0$). We conclude that the dimension of $H^0(\mathbb{P}^r_y, \mathcal{L}_y)$ is 1 for every y. This means, $\mathcal{L}_y \cong \mathcal{O}$, for every $y \in S$.

Step 2. We show that, if $\mathcal{L}_y$ is trivial for every y, then there exists a line bundle $\mathcal{M}$ on S such that $f^*(\mathcal{M}) \cong \mathcal{L}$. The claim is that $\mathcal{M} := f_*(\mathcal{L})$ has the desired property. We have already shown above that $\mathcal{M}$ is locally free of rank $H^0(\mathbb{P}^r_y, \mathcal{O}_y) = 1$. By adjunction, we have a natural map $f^*\mathcal{M} \to \mathcal{L}$. We claim that this is an isomorphism. Since the statement is functorial, we may replace S by smaller open sets and assume that $\mathcal{M}$ is free of rank 1. Choose a generator s for it and let σ be the image of s in $\mathcal{L}$ under the adjunction map $f^*\mathcal{M} \to \mathcal{L}$. We show that σ is a generator for $\mathcal{L}$ by proving that it is nowhere vanishing. This is quite easy, because if σ vanished at a point $x \in \mathbb{P}^r_S$, it would vanish along the entire fiber $\mathbb{P}^r_y$, where $y = f(x)$. But this would then imply that s vanishes at y which is impossible. $\square$

Let us summarize the key facts that made the above proof possible.

- **A**. The map $f\colon \mathbb{P}^r_S \to S$ is flat and proper, so we can use Grothendieck's Base Change and Semicontinuity.
- **B**. For every field k, we have $H^1(\mathbb{P}^r_k, \mathcal{O}) = 0$.
- **C**. For every field k, every line bundle over $\mathbb{P}^r_k$ is of the form $\mathcal{O}(d)$. If for a line bundle $\mathcal{L}$ we have the equality $\dim_k H^0\big(\mathbb{P}^r_k, \mathcal{L}^{\otimes m}\big) = 1$ for every $m \in \mathbb{Z}$, then $\mathcal{L} \cong \mathcal{O}$.

(In fact, in (**A**) knowing the equality with $m = 1$ was already enough to show that $\mathcal{L} \cong \mathcal{O}$. But, for weighted projective spaces this in general will not be enough. More precisely, in *Step 1*, after reducing to the case $\mathcal{L}_y \cong \mathcal{O}$, we will have to repeat the argument for every tensor power $\mathcal{L}^{\otimes m}$.)

All of the above three facts are indeed true for arbitrary weighted projective stacks, as we will see shortly. So, once we have Grothendieck's base change (and Semicontinuity) for stacks, Theorem 4.2 follows (see Appendix I).

Let us verify (**A**), (**B**) and (**C**) for arbitrary weighted projective stacks. It is well knows that $f\colon \mathcal{P}_S(n_0, n_1, \cdots, n_r) \to S$ is proper and flat; this takes care of (**A**). Remark 5.6 implies (**B**). To prove (**C**), observe that $H^0(\mathbb{P}^r_k, \mathcal{O}(d))$ is naturally isomorphic to the k-vector space of homogenous monomials of degree d in weighted variables $\{x_1, \cdots, x_r\}$, where x_i is of weight n_i. Hence, $\dim_k H^0(\mathbb{P}^r_k, \mathcal{O}(d))$ is equal to the number of solutions of the equation

$$a_1 n_0 + a_2 n_1 + \cdots + a_r n_r = d$$

in non-negative integers a_i. It is not in general true that if the number of such solutions for d and d' are equal then $d = d'$. However, it is true that if the number of such solutions are equal for md and md', for all integers m, then $d = d'$. This shows that (**C**) is true, completing our argument.

7. Appendix I: Grothendieck's base change results for algebraic stacks

In this appendix I present a proof, essentially due to Vistoli, of Grothendieck's base change and semicontinuity theorems for the special case of weighted projective stacks.[1]

Proposition 7.1. *Let S be an arbitrary base scheme. Let G be a diagonalizable group scheme (in the sense of* [2]*) over S, and $p\colon X \to S$ a scheme separated over S endowed with a G-action with finite stabilizers. Assume X can be covered by*

[1]Martin Olsson [6] has explained to me that the base change theorem can be proved in full generality using the results of his papers.

invariant open subschemes X_α such that each $p\colon X_\alpha \to S$ is affine. Let $\mathcal{X} = [X/G]$ be the stack quotient and $\mathcal{X}_{mod} = X/G$ the geometric quotient, with $\pi\colon \mathcal{X} \to \mathcal{X}_{mod}$ the moduli map. Let $\pi_*\colon$ $\mathrm{Qcoh}(\mathcal{X}) \to \mathrm{Qcoh}(\mathcal{X}_{mod})$ be the push forward map. Then π_* has the following properties:

- **i.** *π_* is exact and sends flat (over S) quasi-coherent sheaves to flat (over S) quasi-coherent sheaves.*
- **ii.** *Formation of the coarse moduli space commutes with arbitrary base change, and so does π_*. More precisely, for every $S' \to S$, the following diagram is cartesian,*

$$\begin{tikzcd} \mathcal{X}_{S'} \arrow[r, "v"] \arrow[d, "\pi_{S'}"'] & \mathcal{X}_S \arrow[d, "\pi_S"] \\ (\mathcal{X}_{S'})_{mod} \arrow[r, "u"'] & (\mathcal{X}_S)_{mod} \end{tikzcd}$$

and we have $u^* \circ (\pi_S)_* = (\pi_{S'})_* \circ v^*$.

Proof. For each α, the geometric quotient X_α/G is given by $\mathrm{Spec}(p_*\mathcal{O}_{X_\alpha})^G$. That is, $\mathcal{X}_{mod}$ has an affine (relative to S) covering by $\mathrm{Spec}(p_*\mathcal{O}_{X_\alpha})^G$. For any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, viewed as a G-equivariant sheaf on X, its push forward $\pi_*\mathcal{F}$ is given on each $\mathrm{Spec}(p_*\mathcal{O}_{X_\alpha})^G$ by $(p_*\mathcal{F})^G$. The proposition now follows from the fact that, for any diagonalizable group scheme over S, and any G-equivariant quasi-coherent sheaf $\mathcal{A}$ on S, such as $p_*\mathcal{O}_{X_\alpha}$ or $p_*\mathcal{F}$, we have a natural decomposition

$$\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathcal{A}_\lambda,$$

where each $\lambda\colon G \to \mathbb{G}_m$ is a character of G, and Λ is the set of all characters. In particular, for every λ, the functor $\mathcal{A} \mapsto \mathcal{A}_\lambda$ is exact, sends flat (relative to S) quasi-coherent sheaves to flat quasi-coherent sheaves, and commutes with arbitrary base change $S' \to S$. Applying this to our situation, with λ being the trivial character, the proposition follows. □

Corollary 7.2. *If in the above proposition $\mathcal{X}_{mod} \to S$ is proper and S is Noetherian, then Grothendieck's semicontinuity and base change theorems are valid for the morphism $\mathcal{X} \to S$.*

Proof. Push forward everything from $\mathcal{X}$ to $\mathcal{X}_{mod}$ and apply the relevant results for the map $\mathcal{X}_{mod} \to S$. □

Example 7.3. Let $X = \mathbb{A}^r_S - \{0\}$, and consider the weight $(n_0, n_1, \cdots, n_r)$ action of $G = \mathbb{G}_{m,S}$ on X. Let $\{X_i\}_{i=1}^r$ be the standard covering of X, where $X_i = \mathrm{Spec}\,\mathcal{O}_S[x_0, \cdots, x_r, x_i^{-1}]$. Then the conditions of the proposition are satisfied.

References

1. M. Aoki, *Hom stacks*, arXiv:math.AG/0503358, March 2005.
2. M. Demazure and A. Grothendieck, *Schémas en groupes I (SGA 3)*, Lecture notes in mathematcis, No. 151, Springer-Verlag, 1970.
3. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.
4. D. Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin-New York, 1971.

5. G. Laumon and Moret-Bailly L, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer-Verlag, Berlin.
6. M. Olsson, Personal communication.
7. A. Vistoli, Personal communication.