FUNDAMENTAL GROUPS OF ALGEBRAIC STACKS

B. NOOHI

ABSTRACT. We study the fundamental groups of algebraic stacks. We show that these fundamental groups carry an additional structure coming from the inertia groups. We use this additional structure to analyze geometric/ topological properties of stacks. We give an explicit formula for the fundamental group of the coarse moduli space. As an application, we find an explicit formula for the fundamental group of the geometric quotient of an arbitrary algebraic group action. Also, we use these additional structures to give a necessary and sufficient for an algebraic stack to be uniformizable (i.e., quotient of an algebraic space by a finite group action).

1. INTRODUCTION

In [4] Grothendieck introduces a general formalism to associate a *fundamental group* with a pointed scheme X. This is a profinite group which should be thought of as the "profinite completion of the (virtual) Poincaré fundamental group" of X. Grothendieck's ideas are indeed general enough to apply to any reasonable notion of "space" (e.g., a topos). For instance, back to topology, one can apply the theory to (connected locally 1-connected) topological spaces and what comes out is the profinite completion of the usual fundamental group. Grothendieck's theory also applies to Algebraic stacks, for instance via the topos associated to the étale or smooth site of the stack. The aim of this paper is to write out the theory in a rather detailed way, but without using the language of topoi, and then to explore the features that are special to this particular case. For the topos theoretic approach to fundamental groups of algebraic stacks see [12].

Let $\mathcal{X}$ be a connected Algebraic stack. Let x be a geometric point of $\mathcal{X}$, and let ρ_x be its residue gerbe. The following simple observation is essential: *The automorphism group of the gerbe ρ_x is isomorphic to the geometric fundamental group of ρ_x; in particular, we have a natural group homomorphism ω_x from the automorphism group of ρ_x to the fundamental group of $\mathcal{X}$ at x.* This means that the fundamental group of an algebraic stack at a point x comes equipped with an extra structure – namely, the map ω_x. The significance of this map is that it relates local data (the automorphism group of the residue gerbe) to global data (the fundamental group). Since the fundamental groups of $\mathcal{X}$ at different points are isomorphic (with an isomorphism that is unique up to conjugation), we can indeed map the automorphism group of the residue gerbe of any point x' into $\pi_1(\mathcal{X}, x)$. The normal subgroup generated by all the images will be a well-defined subgroup of $\pi_1(\mathcal{X}, x)$. Let N be the closure of this subgroup. We prove the following

Theorem 1.1. *The group $\pi_1(\mathcal{X}, x)/N$ is naturally isomorphic to the fundamental group of the moduli space of $\mathcal{X}$ (see Theorem 7.12 for the precise statement).*

The maps ω_x can be used to detect whether a given algebraic stack is uniformizabe:

Theorem 1.2. *A Deligne-Mumford stack $\mathcal{X}$ is uniformizable if and only if all the maps ω_x are injective (see Theorem 6.2 for the precise statement).*

An algebraic stack being *uniformizable* means that it has a representable finite étale cover by an algebraic space (roughly speaking, its "universal cover" is an algebraic space).

This paper is organized into two parts. Part one consists of the main construction and the main results. Part two should be thought of as a technical companion to part one where we strengthen the results of part one by introducing some more elaborate techniques.

Acknowledgments. Part one of this paper is a modification of some part of my MIT thesis. I would like to express my gratitude to my thesis advisors S. Kleiman and F. Diamond. I learned algebraic stacks in a fantastic course taught by J. de Jong at MIT in spring 1999. I am indebted to him. The second part of the this paper was added during my visit to UBC. I would like to thank Kai Behrend for the invitation and for sharing many of his ideas.

Contents

2. Review and Conventions

Our reference for the theory of stacks is [9]. We quickly review a few basic facts that we will need, some of which are not explicitly mentioned in [9].

We begin with some conventions:

In this paper, whenever we use the word *category* for something that is really a 2-category, we simply mean the category that is obtained by identifying 2-isomorphic 1-morphisms. Typical example: the 2-category of algebraic stacks (say, over a fixed base algebraic space S).

We use calligraphic symbols for algebraic stacks and Roman symbols for algebraic spaces or schemes. We denote the Zariski topological space of an algebraic stack $\mathcal{X}$ by $|\mathcal{X}|$.

Throughout this paper the word *representable* means representable by algebraic spaces. Although it is an extraneous condition in most part of the paper, we will assume that our algebraic stacks are locally Noetherian.

By a smooth (respectively, flat) *chart* for an algebraic stack $\mathcal{X}$ we mean a smooth (respectively, flat and of finite presentation) surjective map $p\colon X \to \mathcal{X}$, where X is an algebraic space. When we do not specify an adjective (flat or smooth) for a chart, we mean a smooth chart. To a smooth (respectively, flat) chart, we can associate a smooth (respectively, flat) groupoid $R \overset{s,t}{\rightrightarrows} X$, where $R := X \times_{\mathcal{X}} X$, and the maps s, t are the two projections. Conversely, given a flat groupoid $R := X \times_{\mathcal{X}} X$ on an algebraic space X, we can construct a quotient stack $\mathcal{X} = [X/R]$ (see Artin's theorem in [9], Section 10). These two constructions are inverse to each other.

A useful fact to keep in mind is the following: Let $X \to \mathcal{X}$ be a chart for $\mathcal{X}$, and let $R_X \rightrightarrows X$ be the corresponding groupoid. Let $\mathcal{Y} \to \mathcal{X}$ be a representable morphism, and let $Y \to \mathcal{Y}$ be the pull-back chart for $\mathcal{Y}$. Let $R_Y \rightrightarrows Y$ be the corresponding groupoid. Then, the diagram

$$(1)\qquad \begin{array}{ccc} R_Y & \longrightarrow & R_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is cartesian, where the vertical arrows are either source or target maps.

A *covering space* for an algebraic stack $\mathcal{X}$ is a representable finite étale morphism $f\colon \mathcal{Y} \to \mathcal{X}$, where $\mathcal{Y}$ is a connected algebraic stack. We sometimes call such a map a *covering map*.

We occasionally use the adjectives *étale* or *finite* for *non-representable* maps. In the former case, the definition is easy as being étale is local on the source. So, to check whether a given map is étale we can pick a chart and replace the source by a scheme, in which case the map becomes representable. The definition of finite is a bit trickier. We say a morphism is finite if it is proper and quasi-finite. (One could also use Chevalley's theorem to define affine morphisms (not necessarily representable), and then define a finite map to be a proper affine map.)

We say a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a *monomorphism*, if for any scheme T, the induced map $\mathcal{X}(T) \to \mathcal{Y}(T)$ on the groupoids of T-points is fully faithful. When f is representable, any base change of f to a map of schemes will be a monomorphism in the usual sense, and vice versa. In particular, a locally closed immersion is a monomorphism.

2.1. **Stabilizer group of an algebraic stack.** Let $R \rightrightarrows X$ be a flat groupoid over an algebraic space X, and let $\mathcal{X} = [X/R]$ be the the corresponding quotient stack. The *stabilizer group* of $R \rightrightarrows X$, or simply of $\mathcal{X}$ (a harmless abuse of terminology!), is the group space $S \to X$ that is defined by the following cartesian diagram:

$$\begin{array}{ccc} S_X & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ X & \xrightarrow[\Delta]{} & X \times X \end{array}$$

or, equivalently, by the following cartesian diagram

$$\begin{array}{ccccc} S_X & \longrightarrow & \mathcal{S}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[p]{} & \mathcal{X} & \xrightarrow[\Delta]{} & \mathcal{X} \times \mathcal{X} \end{array}$$

The group stack $\mathcal{S}_{\mathcal{X}} \to \mathcal{X}$ is called the *inertia stack* or the *stabilizer group stack* or the *automorphism group stack* of $\mathcal{X}$. The group stack $\mathcal{S}_{\mathcal{X}} \to \mathcal{X}$ is representable. In particular, any property of (representable) morphisms of stacks can be attributed to the stabilizer of an algebraic stack (e.g., properties such as finite, quasi-finite, étale, unramified, reduced/connected geometric fibers, and so on). The stabilizer group stack of an algebraic stack $\mathcal{X}$ is always of finite type and separated; when $\mathcal{X}$ is a Deligne-Mumford stack it is unramified (hence quasi-finite) as well. When $x\colon \operatorname{Spec} k \to \mathcal{X}$ is a geometric point, then the *stabilizer group scheme of* x, denoted S_x, is defined to be the fiber of $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ over the diagonal point $x \to \mathcal{X} \times \mathcal{X}$ (equivalently, S_x is the fiber of $\mathcal{S}_{\mathcal{X}} \to \mathcal{X}$ over x). If x_0 is a lift of x to a chart $X \to \mathcal{X}$ for $\mathcal{X}$, then S_x is naturally isomorphic to the fiber of $S_X \to X$ over x_0. We can also talk about the stabilizer group of a point in the underlying space $|\mathcal{X}|$, but it will be defined only up to isomorphism (unless we fix a geometric point for it).

Throughout the text, whenever we use the phrase 'stabilizer group of $\mathcal{X}$', the reader can either think of the stabilizer group stack $\mathcal{S}_{\mathcal{X}} \to \mathcal{X}$, defined as above, or, those who do not like the word "group stack", can implicitly fix a chart and work with the stabilizer group of the corresponding groupoid.

The following two lemmas make it easy to play around with stabilizer groups. Proofs are easy consequences of definitions and are left to the reader.

Lemma 2.1. *Let $\mathcal{X}$ be an algebraic stack, and let $p\colon X \to \mathcal{X}$ be a chart for it. Let $X' \to \mathcal{X}$ be another chart for $\mathcal{X}$ that factors through X. Then, we have the following cartesian diagram of stabilizer group spaces:*

$$\begin{array}{ccc} S_{X'} & \longrightarrow & S_X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

Let $\mathcal{X}$ be an algebraic stack and let $p\colon X \to \mathcal{X}$ be a chart for it. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of stacks, and let $q\colon Y \to \mathcal{Y}$ be the chart for $\mathcal{Y}$ obtained by pulling back p via f. There is a natural group homomorphism $S_Y \to S_X \times_X Y$ (as group spaces over Y). The following lemma says that, as far as algebro-geometric properties of morphisms are concerned, this map behaves like the diagonal map $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$.

Lemma 2.2. *There is a natural cartesian diagram*

$$\begin{array}{ccc} S_Y & \longrightarrow & S_X \times_X Y \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow[\Delta]{} & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \end{array}$$

Corollary 2.3. *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of stacks, and let $q \colon Y \to \mathcal{Y}$ be the chart for $\mathcal{Y}$ obtained by pulling back p via f. Then the natural morphism $\varphi \colon S_Y \to S_X \times_X Y$ is a monomorphism. Furthermore:*

- **i)** *If f is unramified, then φ embeds S_Y as an open subgroup space of $S_X \times_X Y$ (as group spaces over Y). In particular, the stabilizer group of a geometric point y of $\mathcal{Y}$ is isomorphic to an open subgroup of that of $f(y)$. In particular, the stabilizer group of y is reduced if and only if the stabilizer group of $f(y)$ is so. Finally, $\mathcal{Y}$ has reduced stabilizers if and only if $\mathcal{X}$ does.*
- **ii)** *If f is separated, then φ embeds S_Y as a closed subgroup space of $S_X \times_X Y$ (as group spaces over Y). In particular, if $\mathcal{X}$ has finite (respctively, proper) stabilizer, then so does $\mathcal{Y}$.*

Proof. When $f \colon \mathcal{Y} \to \mathcal{X}$ is representable the diagonal $\Delta \colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is a monomorphism (by taking a chart for $\mathcal{X}$ we reduce the statement to the case where $\mathcal{X}$, hence $\mathcal{Y}$, are algebraic spaces, in which case the claim is obvious). If, furthermore, f is unramified (respectively, separated), then Δ is on open (respectively closed) embedding. Now apply Lemma 2.2. ☐

2.2. **Groupoids.** The language of groupoids turns out to the most natural one for formulating our results. In this section we fix some notations and terminology related to groupoids.

Let Π be a groupoid. We use the words *object* and *point* interchangeably to refer to objects of Π (viewed as a category). Similarly, we use the words *morphism* and *arrow* interchangeably to refer to morphisms of Π. For a point $x \in \Pi$, we denote the automorphism group of Π at x by $\Pi(x)$.

Let Π be a groupoid, and let Π' be a subgroupoid of it. We say that Π' is a *normal* subgroupoid, if for every arrow b in Π' and every arrow a in Π, we have $aba^{-1} \in \Pi'$, whenever the composition is defined. For an arbitrary subgroupoid $\Pi' \subseteq \Pi$, we define its *normal closure*, denoted $N\Pi'$, to be the smallest normal subgroup of Π containing Π'.

The above definition of a normal subgroupoid is equivalent to the following: For every pair of points x_1 and x_2 in Π' and every arrow between them, the induced isomorphism $\Pi(x_1) \to \Pi(x_2)$ maps $\Pi'(x_1)$ isomorphically to $\Pi'(x_2)$.

Part 1. Basic results

3. Hidden paths in algebraic stacks

Algebraic stacks form a 2-category, meaning that, given a pair of morphism with the same source and target, we could talk about transformations (or 2-morphisms) between them. These 2-morphisms behave somewhat like *homotopies*. In this section we will exploit this idea systematically. In the next section, we see how to obtain actual homotopies out of these transformations. In this paper we are only interested in the case where the source is the spectrum of an algebraically closed field. We make the following

Definition 3.1. Let $x, x' \colon \operatorname{Spec} k \to \mathcal{X}$ be two geometric points. By a *hidden path* from x to x', denoted $x \rightsquigarrow x'$, we mean a transformation from x to x'. The *hidden fundamental group* of $\mathcal{X}$ at x, denoted $\pi_1^h(\mathcal{X}, x)$, is the group of self-transformations of x. We define the *hidden fundamental groupoid* of $\mathcal{X}$, denoted $\Pi_1^h(\mathcal{X})$, as follows:

- $\operatorname{Ob} \Pi_1^h(\mathcal{X})$={geometric points of $\mathcal{X}$}
- $\operatorname{Mor}(x, x')$={hidden paths $x \rightsquigarrow x'$}.

The multiplication $\gamma_1\gamma_2$ of hidden paths $\gamma_1 \in \operatorname{Mor}(x, x')$ and $\gamma_2 \in \operatorname{Mor}(x', x'')$ is defined by composition of transformations. For any geometric point $x \colon \operatorname{Spec} k \to \mathcal{X}$, the automorphism group of x, viewed as an object of the category $\Pi_1^h(\mathcal{X})$, is equal to the hidden fundamental group $\pi_1^h(\mathcal{X}, x)$.

See Example 4.3 for a (typical) example of a hidden fundamental group and its relation with the actual fundamental group. This example may also serve as a justification for using the term "hidden fundamental group".

The following proposition follows immediately from the definition.

Proposition 3.2. *A monomorphism (see Section 2, just before 2.1) of algebraic stacks induces an isomorphism of hidden fundamental groups.*

Definition 3.3. By a *pointed map* (or *a pointed morphism*) $(f, \phi) \colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$ we mean a pair (f, ϕ), where $f \colon \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks and $\phi \colon x \rightsquigarrow f(y)$ is a hidden path (Definition 3.1).

For a morphism $f \colon \mathcal{Y} \to \mathcal{X}$ of algebraic stacks, we obtain an induced map of groupoids $\Pi_1^h(f) \colon \Pi_1^h(\mathcal{Y}) \to \Pi_1^h(\mathcal{X})$. So, for a pointed map $(f, \phi) \colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$, we obtain a natural map $\pi_1^h(f, \phi)$ between the hidden fundamental groups as follows:

$$\begin{aligned} \pi_1^h(f, \phi) \colon \pi_1^h(\mathcal{Y}, y) &\to \pi_1^h(\mathcal{X}, x) \\ \beta \quad &\mapsto \phi f(\beta) \phi^{-1}. \end{aligned}$$

The following proposition gives a simple description of the hidden fundamental groups.

Proposition 3.4. *Let $p \colon X \to \mathcal{X}$ be a chart for $\mathcal{X}$, and let $x \colon \operatorname{Spec} k \to \mathcal{X}$ be a geometric point of $\mathcal{X}$. Let x_0 be a geometric point of X, and let $\phi \colon x \rightsquigarrow p(x_0)$ be a hidden path (i.e., $(p, \phi) \colon (X, x_0) \to (\mathcal{X}, x)$ is a pointed map) . Then we have a natural isomorphism $S_{x_0} \xrightarrow{\sim} \pi_1^h(\mathcal{X}, x)$, where S_{x_0} denotes the set of k-points of the fiber of the stabilizer group space $S \to X$ at x_0. In particular, $\pi_1^h(\mathcal{X}, x)$ has a natural structure of an algebraic group.*

Proof. By definition of the fiber product in the 2-category of algebraic stacks, a $(\operatorname{Spec} k)$-point of S_{x_0} corresponds to a transformation of functors from $p \circ x_0$ to itself. So, we have a natural isomorphism $\lambda \colon S_{x_0} \to \pi_1^h(\mathcal{X}, p(x_0))$. We define $S_{x_0} \xrightarrow{\sim} \pi_1^h(\mathcal{X}, x)$ by sending $s \in S_{x_0}$ to $\phi\lambda(s)\phi^{-1}$. □

Remark 3.5. As we saw in the proof, this isomorphism does depend on the choice of ϕ. Indeed, if $\phi' \colon x \rightsquigarrow p(x_0)$ is another hidden path, then the two isomorphisms will be conjugate by the element $\gamma = \phi'\phi^{-1} \in \pi_1^h(\mathcal{X}, x)$, that is, $\lambda_{\phi'} = \gamma\lambda_\phi\gamma^{-1}$.

By the above proposition, the hidden fundamental group at a geometric point x is isomorphic to the automorphism group of the residue gerbe at x. If $\mathcal{X}$ is defined as a quotient of a group action on an algebraic space X , then the hidden fundamental group at the point x is isomorphic to the isotropy subgroup of any point in X lying above x. If $\mathcal{X}$ is defined as the moduli space of certain objects, then the hidden fundamental group at a point x is naturally isomorphic to the automorphism group of the object represented by x.

Definition 3.6. We say that a geometric point x of $\mathcal{X}$ (or its image in the underlying space of $\mathcal{X}$) is *unramified*, if $\pi_1^h(\mathcal{X}, x)$ is trivial. We say it is *schematic*, if the residue gerbe at x is a scheme (necessarily of the form $\operatorname{Spec} k$).

A point is schematic if and only if its stabilizer group is trivial. It is unramified if and only if its stabilizer group is connected and zero-dimensional. A point is schematic if and only if it is unramified and its stabilizer is reduced. In particular, these notions are equivalent for Deligne-Mumford stacks, or for algebraic stacks over a field of characteristic zero. However, in general they are not equivalent (Example 3.8 below).

It is worthwhile to keep in mind that a point x being schematic or unramified only depends on the residue gerbe at x.

Corollary 3.7. *Let $\mathcal{X}$ be an algebraic stack.*

i) *If $\mathcal{X}$ has quasi-finite stabilizer, then $\pi_1^h(\mathcal{X}, x)$ is finite for every geometric point x.*

ii) *$\mathcal{X}$ is an algebraic space if and only if all its points are schematic.*

Proof. Immediate. □

Example 3.8. *An algebraic stack all whose points are unramified need not be an algebraic space.* For instance, take the classifying stack of a (non-trivial) connected zero-dimensional group scheme over a field (necessarily of positive characteristic). More explicitly, take an elliptic curve defined over a field of positive characteristic and let it act on itself via Frobenius. Then the quotient stack is an algebraic stack whose underlying set is just a single point and the stabilizer group at this point is isomorphic to the kernel of Frobenius which is a connected finite flat group scheme supported at a single point. In particular, $\pi_1^h(\mathcal{X}, x)$ is trivial by Proposition 3.4.

The following lemma is straightforward.

Lemma 3.9. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is representable, then for every geometric point x of $\mathcal{X}$ the induced map $\pi_1^h(\mathcal{X}, x) \to \pi_1^h(\mathcal{Y}, f(x))$ is injective.*

Proof. Follows from Corollary 2.3. □

The converse of the this Proposition is true in zero characteristic, or when $\mathcal{X}$ is a Deligne-Mumford stack. For a counterexample, take the algebraic stack of Example 3.8 and map it to the ground field.

4. The Galois category of an algebraic stack

In this section, we use Grothendieck's formalism of Galois categories to associate with a pointed connected algebraic stack $(\mathcal{X},x)$ a fundamental group $\pi_1(\mathcal{X},x)$. We refer the reader to [4] for an account of the theory of Galois categories and fundamental functors.

The reader who is familiar with the language of topos theory will immediately realize that the Galois category that we associate to a pointed algebraic stack is nothing but the Galois category of locally constant sheaves in the pointed topos associated to the stack (the choice of topology turns out to be immaterial). Therefore, we obtain the same fundamental groups as we would have obtained via the topos theoretic approach. The advantage of our more explicit approach, is that it makes it clear how the ramification of $\mathcal{X}$ at x induces an extra structure on $\pi_1(\mathcal{X},x)$. More precisely, we will show that there is a natural group homomorphism $\omega_x\colon \pi_1^h(\mathcal{X},x)\to\pi_1(\mathcal{X},x)$. The significance of this map is two-fold: on the one hand, it relates local data to global data, and, on the other hand, it gives a homotopy theoretic meaning to the ramification structure of stacks.

Definition 4.1. We say that a morphism $f\colon \mathcal{Y}\to\mathcal{X}$ of connected algebraic stacks is a *covering map*, or $\mathcal{Y}$ is a covering space of $\mathcal{X}$ (via f), if f is a representable finite étale map.

Let $\mathcal{X}$ be a connected algebraic stack and let $x\colon \operatorname{Spec} k\to\mathcal{X}$ be a geometric point. We define the Galois category $\mathbf{C}_{\mathcal{X}}$ of $\mathcal{X}$ and the fundamental functor F_x associated to the point x as follows:

- $\operatorname{Ob}(\mathbf{C}_{\mathcal{X}})=\left\{\begin{array}{ccl}(\mathcal{Y},f) & \mid & \mathcal{Y}\ \text{ an algebraic stack;}\\ & & f\colon\mathcal{Y}\to\mathcal{X}\ \text{ a covering space.}\end{array}\right\}$
- $\operatorname{Mor}_{\mathbf{C}_{\mathcal{X}}}((\mathcal{Y},f),(\mathcal{Z},g))=\left\{\begin{array}{ccl}(a,\Phi) & \mid & a\colon\mathcal{Y}\to\mathcal{Z}\ \text{ a morphism;}\\ & & \Phi\colon f\Rightarrow g\circ a\ \text{ a 2-morphism.}\end{array}\right\}_{/\sim}$

where $\sim$ is defined by

$$(a,\Phi)\sim(b,\Psi)\ \text{ if }\ \exists\Gamma\colon a\Rightarrow b\ \text{ such that } g(\Gamma)\circ\Phi=\Psi.$$

In other words, $\mathbf{C}_{\mathcal{X}}$ is the 1-category associated to the 2-category of finite étale stacks over $\mathcal{X}$, obtained by declaring the 2-isomorphisms to be identity.
The fundamental functor $F_x\colon \mathbf{C}_{\mathcal{X}}\to((Sets))$ is defined as follows

$$F_x(\mathcal{Y})=\left\{\begin{array}{ccl}(y,\phi) & \mid & y\colon \operatorname{Spec}k\to\mathcal{Y}\ \text{ geometric point}\\ & & \phi\colon x\rightsquigarrow f(y)\ \text{ hidden path}\end{array}\right\}_{/\sim}$$

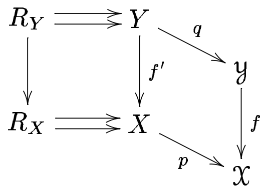
where $\sim$ is defined by

$$(y,\phi)\sim(y',\phi')\ \text{ if }\ \exists\beta\colon y\rightsquigarrow y'\ \text{ such that } f(\beta)\circ\phi=\phi'.$$

Note that the set $F_x(\mathcal{Y})$ is in a natural bijection with the underlying set of the geometric fiber $\operatorname{Spec}k\times_{\mathcal{X}}\mathcal{Y}$, which is isomorphic to a disjoint union of copies of $\operatorname{Spec}k$.

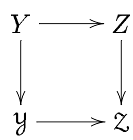
Theorem 4.2. *The pair* $(\mathbf{C}_{\mathcal{X}}, F_x)$ *forms a Galois category.*

Proof. We verify the axioms $G1, ..., G6$ of ([4], page 118). We drop the subscripts and write $(\mathbf{C}, F)$ for $(\mathbf{C}_{\mathcal{X}}, F_x)$.
Axiom $G1$. Obvious.
Axiom $G2$. Direct sums (i.e., disjoint unions) obviously exist. Let G be a finite group of automorphisms of $\mathcal{Y} \in \mathbf{C}$; i.e. G is a subgroup of $\mathrm{Aut}_{\mathbf{C}}(\mathcal{Y})$. We want to show that there exists a quotient $\mathcal{Y}/G \in \mathbf{C}$ for this action. Let $p\colon X \to \mathcal{X}$ be a chart for $\mathcal{X}$, and let $R_X \rightrightarrows X$ be the corresponding groupoid. Set $Y := X \times_{\mathcal{X}} \mathcal{Y}$. Then the second projection makes $q\colon Y \to \mathcal{Y}$ a chart for $\mathcal{Y}$. Let $R_Y \rightrightarrows Y$ be the corresponding groupoid. Then we have the following diagram in which all sub-squares are cartesian:



If (a, Φ) is an automorphism of $\mathcal{Y}$, then, by the definition of 2-product, it induces an automorphism of Y relative to X. Similarly, we get an automorphism of R_Y relative to R_X. These automorphisms respect the structure maps of the groupoid. On the other hand, if $(a, \Phi) \sim (a', \Phi')$, then the induced maps are the same. Therefore, if G is a (finite) group of automorphisms of $\mathcal{Y}$, then we have an action of G on Y and on R_Y (relative to X and R_X, respectively) in a way that the structure maps of the groupoid are G-equivariant. Now set $Z := Y/G$ and $R_Z := R_Y/G$, with the induced groupoid structure $R_Z \rightrightarrows Z$. The stack $\mathcal{Z} := [Z/R_Z]$ with induced map to $\mathcal{X}$ is easily checked to have the desired property.
Axiom $G3$. Obvious, because any morphism in $\mathbf{C}$ is finite étale.
Axiom $G4$. Left exactness follows from the fact that F commutes with arbitrary fiber products.
Axiom $G5$. It is easy to see that F commutes with disjoint unions and sends epimorphisms to epimorphisms. Let G be a finite group of automorphisms of $\mathcal{Y} \in \mathbf{C}$. Let $\mathcal{Z} = \mathcal{Y}/G$, as constructed in $G2$. To prove that $F(\mathcal{Y})/G \to F(\mathcal{Z})$ is bijective, first note that it is surjective, because $\mathcal{Y} \to \mathcal{Z}$ is an epimorphism by construction (see Axiom G2).

Now let y_1 and y_2 be two geometric points of $\mathcal{Y}$ that map to the same point z in $\mathcal{Z}$. We need to show that there is an element $g \in G$ such that $g(y_1) = y_2$. By the construction of $\mathcal{Z}$ in $G2$ above, we have the following cartesian diagram:



Let z' be an arbitrary point above z in Z, and let y'_1 and y'_2 be the points above y_1 and y_2 that map to z. Then, since $Z = Y/G$ (see $G2$), there exists a $g \in G$ such that $g(y'_1) = y'_2$. We obviously have $g(y_1) = y_2$.
Axiom $G6$. Left to the reader. $\square$

Following [4], we define $\pi_1(\mathcal{X},x)$ to be the group of self-transformations of the functor F. The group $\pi_1(\mathcal{X},x)$ is a profinite group. Recall that, by definition, a profinite group is a compact Hausdorff totally disconnected topological group. Equivalently, a profinite group is a group that is isomorphic to a directed inverse limit of finite groups. There is an equivalence of categories between $\mathbf{C}$ and the category of $\pi_1(\mathcal{X},x)$-sets, under which F correspond to the forgetful functor. If x' is another geometric point, then $\pi_1(\mathcal{X},x')$ is isomorphic to $\pi_1(\mathcal{X},x)$ via an isomorphism that is unique up to conjugation by an element of $\pi_1(\mathcal{X},x')$. The fundamental group $\pi_1(\mathcal{X},x)$ defined above classifies pointed covering spaces of $(\mathcal{X},x)$, in the sense that, there is a one-to-one correspondence between isomorphism classes of pointed connected covering spaces of $(\mathcal{X},x)$ and open subgroups of $\pi_1(\mathcal{X},x)$.

We define the fundamental groupoid $\Pi_1(\mathcal{X})$ of $\mathcal{X}$ as follows:

- $\mathrm{Ob}\,\Pi_1(\mathcal{X})=\{$geometric points of $\mathcal{X}\}$
- $\mathrm{Mor}(x,x')=\{$transformations of functors $F_x \to F_{x'}\}$

The multiplication $\gamma_1\gamma_2$ of "paths" $\gamma_1 \in \mathrm{Mor}(x,x')$ and $\gamma_2 \in \mathrm{Mor}(x',x'')$ is defined to be the composition of transformations. The fundamental groupoid is a connected groupoid whose group of automorphisms at any point $x \in \Pi_1(\mathcal{X})$ is $\pi_1(\mathcal{X},x)$.

Let $f\colon \mathcal{Y} \to \mathcal{X}$ be an arbitrary covering space, and let $x\colon \mathrm{Spec}\,k \to \mathcal{X}$ and $x'\colon \mathrm{Spec}\,k \to \mathcal{X}$ be two geometric points. Let $\gamma\colon x \rightsquigarrow x'$ be a hidden path. Then we obtain a map of sets $F_x(\mathcal{Y}) \to F_{x'}(\mathcal{Y})$ by sending (y,ϕ) to $(y,\gamma^{-1}\phi)$. This map is functorial in $\mathcal{Y}$; so it gives rise to a transformation of functors $F_x \to F_{x'}$, that is, an element in $\mathrm{Mor}_{\Pi_1(\mathcal{X})}(x,x')$. This construction produces a natural map of groupoids $\Omega\colon \Pi_1^h(\mathcal{X}) \to \Pi_1(\mathcal{X})$. In particular, for any geometric point x, we have a natural group homomorphism $\omega_x\colon \pi_1^h(\mathcal{X},x) \to \pi_1(\mathcal{X},x)$. These constructions are all functorial in the following sense. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks. Then we have natural maps of groupoids $\Pi_1(f)\colon \Pi_1(\mathcal{Y}) \to \Pi_1(\mathcal{X})$ and $\Pi_1^h(f)\colon \Pi_1^h(\mathcal{Y}) \to \Pi_1^h(\mathcal{X})$ that are compatible with Ω, i.e., $\Pi_1(f)\circ\Omega_{\mathcal{Y}} = \Omega_{\mathcal{X}}\circ\Pi_1^h(f)$. Similarly, if $(f,\phi)\colon(\mathcal{Y},y)\to(\mathcal{X},x)$ is a pointed map, we obtain natural maps $\pi_1(f,\phi)\colon \pi_1(\mathcal{Y},y)\to\pi_1(\mathcal{X},x)$ and $\pi_1^h(f,\phi)\colon \pi_1^h(\mathcal{Y},y)\to\pi_1^h(\mathcal{X},x)$. More explicitly, for $\beta\in\pi_1(\mathcal{Y},y)$, $\pi_1(f,\phi)(\beta)$ is defined to be $\phi\pi_1(f)(\beta)\phi^{-1}$. The map $\pi_1^h(f,\phi)\colon \pi_1^h(\mathcal{Y},y)\to\pi_1^h(\mathcal{X},x)$, whose definition is similar to that of $\pi_1(f,\phi)$, has already been introduced in the previous section. Once again we have the compatibility relation $\pi_1(f,\phi)\circ\omega_y = \omega_x\circ\pi_1^h(f,\phi)$.

Example 4.3. Let G be an algebraic group of finite type over an algebraically closed field k. Let $X=\mathrm{Spec}\,k$, and let $\mathcal{X}=[X/G]$ be the quotient of X under the trivial action of G (i.e., the classifying stack of G) made into a pointed stack via the quotient map $\mathrm{Spec}\,k\to\mathcal{X}$. Then, $\pi_1(\mathcal{X},x)$ is naturally isomorphic to the group G/G^0 of connected components of G, and $\pi_1^h(\mathcal{X},x)$ is naturally isomorphic to G itself (more precisely, the group of k-points of G). The map ω_x is simply the quotient map $G\to G/G^0$. Proof that $\pi_1(\mathcal{X},x)\cong G/G^0$ is easy. For instance, we could use a fiber homotopy exact sequence argument (see the appendix) to prove the result. We give a more direct proof: We have a Galois covering $[\mathrm{Spec}\,k/G^0]\to\mathcal{X}$ whose Galois group is G/G^0. The claim follows if we show that $[\mathrm{Spec}\,k/G^0]$ is simply connected. So we may assume G is connected. Let $\mathcal{Y}\to\mathcal{X}$ be a covering map, $Y\to\mathcal{Y}$ the pull-back of $X\to\mathcal{X}$, and $R_Y\rightrightarrows Y$ the corresponding groupoid. Then, Y, being a covering of $X=\mathrm{Spec}\,k$, is a disjoint union of copies of X. It follows from Corollary 2.3, that the diagram

$$\begin{array}{ccc} S_Y & \longrightarrow & S_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is cartesian (because the natural map $S_Y \to S_X \times_X Y$ is both open and closed embedding and $S_X = G$ is connected). Note that $R_X = S_X$, so it follows from the cartesian diagram (1) of Section 2, that S_Y is equal to R_Y. Therefore, $\mathcal{Y}$ is just a disjoint union of copies of $\mathcal{X}$, hence the claim. (Note that in this example we can replace $X = \operatorname{Spec} k$ by any simply connected X.)

5. Basic properties of hidden fundamental groups

In this section we prove a few basic facts about hidden fundamental groups. The following lemma is essential.

Lemma 5.1. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks. Let y be a geometric point of $\mathcal{Y}$, and let $x = f(y)$ be its image in $\mathcal{X}$. Consider the following commutative diagram, obtained from the functoriality of the hidden fundamental groups:*

$$\begin{array}{ccc} \pi_1^h(\mathcal{Y}, y) & \xrightarrow{\omega_y} & \pi_1(\mathcal{Y}, y) \\ {\scriptstyle \pi_1^h(f)}\downarrow & & \downarrow{\scriptstyle \pi_1(f)} \\ \pi_1^h(\mathcal{X}, x) & \xrightarrow[\omega_x]{} & \pi_1(\mathcal{X}, x) \end{array}$$

If f is a covering map, then this diagram is cartesian.

Proof. Let γ be in $\pi_1^h(\mathcal{X}, x)$. We want to show that, if $\omega_x(\gamma)$ is in the image of $\pi_1(f)$, then there exists a unique $\alpha \in \pi_1^h(\mathcal{Y}, y)$ that maps to γ via $\pi_1^h(f)$. The uniqueness is obvious, since $\pi_1^h(f)$ is injective by Lemma 3.9. On the other hand, $\omega_x(\gamma)$ being in the image of $\pi_1(f)$ exactly means that, under the action of γ on $F_x(\mathcal{Y})$, the point (y, id) remains invariant. That means, $(y, id) \sim (y, \gamma^{-1})$. Therefore, there exists $\beta \in \pi_1^h(\mathcal{Y}, y)$ such that $f(\beta) = \gamma^{-1}$. The element $\alpha = \beta^{-1}$ has the desired property. □

Remark 5.2. The above lemma fails when f is not representable. For instance, let $\mathcal{Y}$ be the classifying space of an arbitrary non-trivial finite group, and let f be the moduli map. Then ω_x is zero, but ω_y is not.

Corollary 5.3. *The kernel of ω_x is equal to $\bigcap \operatorname{im}(\pi_1^h(f, \phi))$, where the intersection is taken over all pointed covering spaces $(f, \phi)\colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$.*

Proof. By Lemma 5.1 we have,

$$\begin{aligned} \bigcap \operatorname{im}(\pi_1^h(f, \phi)) &= \bigcap \omega_x^{-1}(\operatorname{im}(\pi_1(f, \phi))) \\ &= \omega_x^{-1}(\bigcap \operatorname{im}(\pi_1(f, \phi))) \\ &= \omega_x^{-1}(\{1\}) = \ker \omega_x. \end{aligned}$$

□

Corollary 5.4. *Let $\mathcal{X}$ be an algebraic stack, and let x be a geometric point. Then the map $\omega_x \colon \pi_1^h(\mathcal{X}, x) \to \pi_1(\mathcal{X}, x)$ has a finite image. Furthermore, the following conditions are equivalent:*

i) *The map ω_x is injective (respectively, ω_x is injective and x has a reduced stabilizer).*

ii) *There exists a covering space $f \colon \mathcal{Y} \to \mathcal{X}$ with an unramified (respectively, a schematic) point y of $\mathcal{Y}$ lying above x.*

Furthermore, if $f \colon \mathcal{Y} \to \mathcal{X}$ of part (**ii**) *is Galois, then every point in $\mathcal{Y}$ lying above x is unramified (respectively, schematic).*

Proof. That ω_x has finite image follows from the fact that ω_x factors through the fundamental group of the residue gerbe at x, which is finite (example 4.3). The equivalence of (**i**) and (**ii**) follows from Corollary 5.3 and Corollary 2.3, plus the fact that ω_x has finite image. The last statement is obvious. □

To prove the next lemma we quote a result from EGA.

Proposition 5.5 ([5] , 18.12.7)**.** *Let $f \colon X \to Y$ be a morphism of schemes that is locally of finite type, and let y be a point in Y. In order for y to have an open neighborhood U such that the restriction map $f \colon f^{-1}(U) \to U$ is a closed immersion it is necessary and sufficient that the fiber X_y be either empty or isomorphic to a copy of y, and that there exist an open V containing y such that the restriction map $f \colon f^{-1}(V) \to V$ be universally closed.*

Lemma 5.6. *If $\mathcal{X}$ has finite stabilizer, then the set of all schematic points of $\mathcal{X}$ is open. This set is the underlying set of the largest open substack of $\mathcal{X}$ that is an algebraic space.*

Proof. The first assertion follows from Proposition 5.5 applied to the stabilizer group scheme (relative to some chart). The second assertion follows from Corollary 3.7(**ii**). □

Remark 5.7. In fact, the above result is still valid if we assume the stabilizer is only universally closed (e.g., proper).

Corollary 5.8. *Let $\mathcal{X}$ be an algebraic stack with finite stabilizer, and let $f \colon \mathcal{Y} \to \mathcal{X}$ be a representable finite étale cover. Let x be a point in $\mathcal{X}$ so that every points in $\mathcal{Y}$ lying above x is schematic. Then, there exists an open neighborhood U of x so that the inverse image of U in $\mathcal{Y}$ consists entirely of schematic points (i.e., is an algebraic space).*

Proof. By Corollary 2.3 (**ii**), $\mathcal{Y}$ has finite stabilizer. Hence, by Lemma 5.6, there exists an open set $V \subseteq \mathcal{Y}$ containing the fiber of x all of whose points are schematic. Since f is closed, we could find an open $U \subseteq \mathcal{X}$ whose inverse image is contained in V (say, the complement of $f(\mathcal{Y} \backslash V)$). The open set U has the desired property. □

Example 5.9. *Counterexample when stabilizer is not finite.* Take the affine line $\mathbb{A}^1$ and let the group scheme $\mathbb{A}^1 \coprod \mathbb{A}^1 \backslash \{0\} \to X$ act (trivially) on it. The resulting quotient stack, call it $\mathcal{X}$, has only one schematic point. An alternative way of constructing $\mathcal{X}$ is to mode out the affine line with double origin by the action of $\mathbb{Z}/2\mathbb{Z}$ that leaves all the points invariant and swaps the two origins. Hence, $\mathcal{X}$ is uniformizable and we have $\pi_1(\mathcal{X}) \cong \mathbb{Z}/2\mathbb{Z}$. By Corollary 5.4, all the maps ω_x are isomorphisms, except for the one at the unique schematic point ("the origin") for which ω_x is the zero map.

Example 5.10. *An algebraic stack with proper stabilizer where the set of unramified points is not open.* Let k be a field of characteristic 2. Let $\mathbb{A}^1 = \operatorname{Spec} k[z]$ be the affine line, and let $G := \operatorname{Spec} k[x,y]/(y^2 = yx)$. Then it is easy to see that G is a finite flat scheme over $\mathbb{A}^1$ via the structure map

$$\operatorname{Spec} k[x,y]/(y^2 = yx) \to \operatorname{Spec} k[x].$$

A bit less non-trivial is the fact that G is indeed a group scheme over $\mathbb{A}^1$. This can be seen as follows. An affine scheme T over $\mathbb{A}^1$ is determined by a pair (R, r) consisting of a k-algebra R and an element $r \in R$; a T-point of G over $\mathbb{A}^1$ is determined by an element $a \in R$ that satisfies the equation $a^2 = ar$. This set is naturally an Abelian group under addition. Therefore, G is naturally a group scheme over $\mathbb{A}^1$. This group scheme is a constant group scheme outside the origin, but is non-reduced over the origin. The quotient of the trivial action of G on $\mathbb{A}^1$ is an algebraic stack that has exactly one unramified point.

Example 5.11. *A Deligne-Mumford stack with finite étale stabilizer where ω_x is never injective.* Let $\mathcal{X}$ be the $\mathbb{Z}/2\mathbb{Z}$-gerbe over $\mathbb{P}^1$ associated to the nontrivial class in $H^2(\mathbb{P}^1, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. For all points $x \in \mathcal{X}$, we have $\pi_1^h(\mathcal{X}, x) \cong \mathbb{Z}/2\mathbb{Z}$. We prove in Section 9 that $\pi_1(\mathcal{X}, x)$ is trivial. Therefore, ω_x is the zero map for all points x.

Examples 5.10 and 5.11 are simply connected (when the base field is algebraically closed). For Example 5.10 we can use an argument as in Example 4.3. The proof that Example 5.11 is simply connected can be found in Section 9.

6. Classification of uniformizable stacks

The main result of this section is Theorem 6.2 which gives a necessary and sufficient condition for the "universal cover" of an algebraic stack to be an algebraic space.

Definition 6.1. We say that an algebraic stack $\mathcal{X}$ is *uniformizable* if it has a covering space that is an algebraic space.

By ([9], Theorem 6.1), $\mathcal{X}$ is uniformizable if and only if it is of the form $[X/G]$, where X is an algebraic space and G is a finite group acting on it.

Theorem 6.2. *Let $\mathcal{X}$ be a Noetherian algebraic stack. Then, $\mathcal{X}$ is uniformizable if and only if it is Deligne-Mumford and ω_x is injective for any geometric point x.*

Proof. Necessity follows from Corollary 5.4. Let us prove the sufficiency. Since the stabilizer of $\mathcal{X}$ is quasi-finite, there exists a stratification $\emptyset = \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_n = \mathcal{X}$ by closed substacks such that the stabilizer of each $\mathcal{Z}_i = \mathcal{X}_i \backslash \mathcal{X}_{i-1}$ is finite. Consider $\mathcal{Z}_i$ for some i, and let x be a geometric point in $\mathcal{Z}_i$. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a Galois cover of $\mathcal{X}$ as in Corollary 5.4 (note that we can always replace an arbitrary cover by a Galois cover that dominates it), and let $V_x \subseteq \pi_1(\mathcal{X}, x)$ be the corresponding open normal subgroup (of finite index). By Corollary 5.8, there is an open neighborhood of V_x of x in $\mathcal{Z}_i$ so that the inverse image of U_x in $\mathcal{Y}$ consists entirely of schematic points. The V_x, for various x in $\mathcal{Z}_i$, form an open covering of $\mathcal{Z}$, so, by Noetherian hypothesis, we can find finitely many of V_x that still cover

$\mathcal{Z}_i$. Let $U_i \subseteq \pi_1(\mathcal{X}, x)$ be the intersection of the corresponding subgroups. Then, $U_i \subseteq \pi_1(\mathcal{X}, x)$ is an open normal subgroup (of finite index). Furthermore, the Galois cover associated to U_i, restricted to $\mathcal{Z}_i$, consists entirely of schematic points. Now set $U := U_1 \cap U_2 \cap \cdots \cap U_n$. The Galois cover associated to U consists entirely of schematic points, hence is an algebraic space by Corollary 3.7 (**ii**). □

Example 6.3. Let $\mathcal{C}$ be a (not necessarily compact) smooth Deligne-Mumford curve over $\mathbb{C}$ whose moduli space (necessarily a smooth algebraic curve) is not isomorphic to $\mathbb{P}^1$, or, if it is, either $\mathcal{C}$ has at least three corner points (i.e., points at which the stabilizer group jumps), or it has two corner points with isomorphic stabilizer groups. In this case, an explicit calculation of the fundamental groups using van Kampen's theorem shows that all the maps ω_x are injective. Therefore, the universal cover of $\mathcal{C}$ is either $\mathbb{C}$, $\mathbb{P}^1$ or the upper half-plane. For a complete classification of smooth Deligne-Mumford stack with their uniformization types and an explicit calculation of their fundamental groups and maps ω_x we refer the reader to [2].

7. Fundamental group of the moduli space

Morally, a *moduli space* for an algebraic stack $\mathcal{X}$ is an algebraic space X_{mod} that "best" approximates the given algebraic stack. To give a more mathematical sense to this statement, we would like to have a morphism $\pi\colon \mathcal{X} \to X_{mod}$ and we impose a list conditions on our morphism f to ensure X_{mod} is "close enough" to $\mathcal{X}$ in a reasonable sense. Depending on what conditions we impose on f, there is a name for the type of moduli space we get. To name a few: categorical quotient, uniform categorical quotient, topological quotient, geometric quotient moduli, coarse moduli space, GC quotient, etc.. The reader can consult [10], [6], [8] for precise definitions. There is a plentiful of results in the literature in the form: *For a certain type of algebraic stacks, a certain type of moduli space exists.* In order not to get bogged down with unnecessary hypotheses, in this paper we will not stick to any of these standard definitions. Instead, we list a few properties that we will need our moduli space to have to make the proofs work, and we assume we are working in a *suitable* category of algebraic stacks in which such moduli spaces exist (see Definition 7.1 below). It turn out that basically any reasonable definition of a moduli spac satisfies our properties. For this reason, in this work, we will use the term *moduli space* without any adjective. We leave it to the reader to check whether his/her favorite definition satisfies our axioms or not.

Vaguely speaking, by a *suitable category* $\mathfrak{C}$ of algebraic stacks we mean a category of algebraic stacks in which a notion of *moduli space* is specified. Once such a category is fixed, when we say an algebraic stack $\mathcal{X}$ has a moduli space we mean $\mathcal{X} \in \mathfrak{C}$. More precisely:

Definition 7.1. By a *suitable* category $\mathfrak{C}$ of algebraic stacks, we mean a subcategory of the category of algebraic stacks that contains the category of algebraic spaces, and in which for every algebraic stack $\mathcal{X}$ we have chosen an algebraic space X_{mod} (called the *moduli space* of $\mathcal{X}$), and a morphism $\pi_{\mathcal{X}}\colon \mathcal{X} \to X_{mod}$ (called the *moduli map*), that satisfy the following axioms:

M1. *Functoriality.* If $f\colon \mathcal{Y} \to \mathcal{X}$ is (the 2-isomorphism class of) a morphism of stacks, then we have a morphism of moduli spaces $f_{mod}\colon Y_{mod} \to X_{mod}$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \pi_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{X}}} \\ Y_{mod} & \xrightarrow[f_{mod}]{} & X_{mod} \end{array}$$

with the usual functoriality properties (i.e., $\mathcal{X} \mapsto X_{mod}$ is a functor and π is a natural transformation between the identity functor and the moduli functor).

M2. *Geometric points.* For any algebraically closed filed k, $\pi_{\mathcal{X}}$ induces a bijection between k-points (up to 2-isomorphism) of $\mathcal{X}$ and k-points of X_{mod}.

M3. *Covering spaces.* If $\mathcal{X}$ is in $\mathfrak{C}$, then so is every covering space of $\mathcal{X}$.

M4. *Invariance under finite étale base change.* If we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{X}}} \\ Y & \xrightarrow[g]{} & X_{mod}, \end{array}$$

in which g is finite étale, then Y is the moduli space of $\mathcal{Y}$ with π its moduli map. Furthermore, g coincides with the induced map of moduli spaces as in **M1**.

M5. *Free quotient of a moduli map is a moduli map.* Let G be a finite group acting freely (see Definition 7.2 below) on $\mathcal{Y}$, and assume $\mathcal{Y}/G$ is in $\mathfrak{C}$. Then in the following (necessarily cartesian) diagram, $\pi_{\mathcal{Y}}/G$ is a moduli map, and the quotient map q is the same as the induced map in **M1**.

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{Y}/G \\ {\scriptstyle \pi_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{Y}}/G} \\ Y_{mod} & \xrightarrow[g]{} & Y_{mod}/G. \end{array}$$

Convention. Throughout this paper, whenever we talk about the moduli space of an algebraic stack $\mathcal{X}$, it is assumed that $\mathcal{X}$ belongs to a certain fixed suitable category as defined above.

Definition 7.2. Let G be a (discrete) finite group acting (via morphisms-up-to-2-isomorphism) on an algebraic stack $\mathcal{X}$, and let x be a geometric point. We define the *stabilizer group* G_x of x to be the set of all $g \in G$ such that $g(x)$ is 2-isomorphic to x. We say that the action is *free*, if the stabilizer group of every geometric point x is trivial.

If $\mathcal{X}$ has a moduli space that satisfies **M1** and **M2**, then the stabilizer group of a point is equal to the usual stabilizer group of the corresponding point in X_{mod} under the induced action of G. Therefore, in this case, the action being free means that the induced action of G on X_{mod} is free.

A caveat about the above definition is in order. Unlike the case of algebraic spaces, the Galois group of a (representable) Galois cover $\mathcal{Y} \to \mathcal{X}$ may *not* act freely on $\mathcal{Y}$ (although it acts freely on the fibers $F_x(\mathcal{Y})$).

Lemma 7.3. *Let $\mathcal{X}$ be an algebraic stack, and let $f: \mathcal{Y} \to \mathcal{X}$ be a Galois cover with Galois group G. Let y be a geometric point of $\mathcal{Y}$ and $x = f(y)$ its image in $\mathcal{X}$. Then we have a short exact sequence*

$$1 \to \pi_1^h(\mathcal{Y}, y) \to \pi_1^h(\mathcal{X}, x) \to G_y \to 1.$$

Proof. Exactness on the left follows from Lemma 3.9. Since $\mathcal{Y} \to \mathcal{X}$ is Galois, the action of an element $\gamma \in \pi_1^h(\mathcal{X}, x)$ (more precisely, its image in $\pi_1(\mathcal{X}, x)$ via ω_x) on the fiber extends to an element $g_\gamma \in G$. Recall from Section 4 that this action takes $(y, id) \in F_x(\mathcal{Y})$ to (y, γ^{-1}) (here by id we mean the constant transformation from x to itself). Therefore, g_γ indeed belongs to G_y. This defines a natural group homomorphism $\pi_1^h(\mathcal{X}, x) \to G_y$. Proof of surjectivity is a easy, but a bit intricate. We need to get down to precise definitions. Recall from the definition of the Galois category $\mathbf{C}_{\mathcal{X}}$ (Section 4 that an element in $g \in G$ can be represented by a pair (a, Φ), where $\Phi: f \Rightarrow f \circ a$. This element belongs to G_y if and only if there is a 2-isomorphism $\alpha: y \rightsquigarrow a(y)$. The effect of (a, Φ) it to take $(y, id) \in F_x(\mathcal{Y})$ to $(a(y), \phi) \sim (y, \phi f(\alpha^{-1})) \in F_x(\mathcal{Y})$, where $\phi: x \rightsquigarrow f(a(y))$ is the transformation induced by Φ. We claim that $\gamma = (\phi f(\alpha^{-1}))^{-1} \in \pi_1^h(\mathcal{X}, x)$ maps to g, i.e., we have $g = g_\gamma$. But we just verified that these two cover automorphisms have the same effect on $(y, id) \in F_x(\mathcal{Y})$, so they must be equal. This proves surjectivity. Exactness in the middle follows from a similar 'hidden path chasing' argument. □

Proposition 7.4. *Let $\mathcal{X}$ be an algebraic stack, and let $f: \mathcal{Y} \to \mathcal{X}$ be a Galois cover with Galois group G. Then the following are equivalent:*

- **i)** *The action of G on $\mathcal{Y}$ is free (see Definition 7.2).*
- **ii)** *For any geometric point y, the induced map $\pi_1^h(\mathcal{Y}, y) \to \pi_1^h(\mathcal{X}, f(y))$ is an isomorphism. Equivalently, the induced map of of groupoids $\Pi_1^h(\mathcal{Y}) \to \Pi_1^h(\mathcal{X})$ is fully faithful.*
- **iii)** *Let $X \to \mathcal{X}$ be a chart for $\mathcal{X}$ and $Y \to \mathcal{Y}$ the pull back chart for $\mathcal{Y}$. Then the morphism $S_Y \to S_X \times_X Y$ of Lemma 2.2 is an isomorphism. Equivalently, the pull back of the stabilizer group stack of $\mathcal{X}$ is the stabilizer group stack of $\mathcal{Y}$ (see Section 2.1).*

Proof. The equivalence of (**i**) and (**ii**) is immediate from Lemma 7.3. To prove the equivalence of (**ii**) and (**iii**), note that, by Corollary 2.3, the map $S_Y \to S_X \times_X Y$ is an open and closed immersion. The equivalence follows immediately from Proposition 3.4. □

In fact, the equivalence of (**ii**) and (**iii**) is true for any covering map $f: \mathcal{Y} \to \mathcal{X}$ (not necessarily Galois).

Definition 7.5. When the two equivalent conditions alluded to in the previous paragraph are satisfied for a covering map f, we say that f is a *fixed point reflecting* (or FPR) morphism.

The following lemma provides a source of FPR morphisms.

Lemma 7.6. *Let $\pi: \mathcal{X} \to A$ be an arbitrary map to an algebraic space, and let $f: B \to A$ be an arbitrary map of algebraic spaces. Then, the induced map $\mathcal{Y} := \mathcal{X} \times_A B \to \mathcal{X}$ is FPR.*

Proof. Let $p\colon X \to \mathcal{X}$ be a chart for $\mathcal{X}$ and $Y := X\times_{\mathcal{X}}\mathcal{Y} = X\times_A B$ the corresponding pull back chart for $\mathcal{Y}$. It follows from the definition of the stabilizer group that the digram

$$\begin{array}{ccc} S_Y & \longrightarrow & S_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is cartesian. The lemma follows now from Proposition 3.4. □

Proposition 7.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks with $\mathcal{X}$ connected. Let $f : \mathcal{Y} \to \mathcal{X}$ be an FPR covering space. Then, the induced map on the moduli spaces $f_{mod}\colon Y_{mod} \to X_{mod}$ is finite étale. Furthermore, the following diagram is cartesian:*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ Y_{mod} & \xrightarrow[f_{mod}]{} & X_{mod} \end{array}$$

Proof. We will only need the result when f is Galois, in which case the claim is immediate from **M5**. In the general (not necessarily Galois) case, take a Galois cover $\mathcal{Y}' \to \mathcal{X}$ that factors through $\mathcal{Y} \to \mathcal{X}$, and use the fact that $\mathcal{Y}' \to \mathcal{Y}$ is also Galois. □

Proposition 7.8. *Let $\mathcal{X}$ be an algebraic stack. Then, we have a one-to-one correspondence*

$$\left\{ \textit{Fixed point reflecting covering spaces of } \mathcal{X}. \right\} \underset{\substack{\textbf{base extension}\\ \textbf{via } \pi_{mod}}}{\overset{\textbf{moduli}}{\rightleftarrows}} \left\{ \textit{Covering spaces of } X_{mod}. \right\}.$$

Similar statement is true for connected covering spaces. Finally, the lemma remains valid if we assume everything pointed.

Proof. The assertion is an immediate consequence of Lemma 7.6 and Lemma 7.7, plus the axiom **M4**. Also note that in a diagram as in Lemma 7.7, $\mathcal{Y}$ is connected if and only if Y_{mod} is so. The pointed version of the lemma is true because a moduli map induces a bijection on the geometric points (**M2**). □

Remark 7.9. We can drop the finiteness condition from both sides of the above correspondence and the result will still be valid. In fact, one could state the correspondence as an equivalence between the category of FPR representable étale maps to $\mathcal{X}$ and the category of representable étale maps to X_{mod}. Note that the the former category always exists, regardless of the existence of X_{mod}. One may interpret this as saying that, even though X_{mod} may not in general exist as an algebraic space, but it always exists as a topos.

Lemma 7.10. *Let G be a profinite group and $H \subseteq G$ a normal subgroup. Then, the intersection of all open normal subgroups of G that contain H is equal to $\bar{H}$. (Here, $\bar{H}$ stands for the closure of H.)*

Proof. The intersection obviously contains $\bar{H}$. To prove the equality, we may assume H is closed. Take an element $g \in G \backslash H$. We want to find an open normal subgroup of G that contains H but not g. Since H is closed, and since open normal subgroups of G form a fundamental system of neighborhoods at the origin, we can find an open normal subgroup $U \subset G$ such that $gU \cap H$ is empty. The group HU is easily seen to have the desired property. □

Lemma 7.11. *Let $\mu\colon G_1 \to G_2$ be a continuous homomorphism of profinite groups. Let $N \subseteq G_1$ be a closed normal subgroup. Assume μ^{-1} induces a one-to-one correspondence between open subgroups of G_2 and those open subgroups of G_1 that contain N. Then, μ induces an isomorphism $\bar{\mu}\colon G_1/N \xrightarrow{\sim} G_2$.*

Proof. The intersection of all open normal subgroups of G_2 is $\{1\}$; so, by the above correspondence, the intersection of all open normal subgroups of G_1 that contain N is equal to $\mu^{-1}(\{1\})$. So Lemma 7.10 implies that $\mu^{-1}(\{1\}) = N$. Hence, $\bar{\mu}$ is injective. To prove the surjectivity, consider the closed subgroup $H := \mu(G_1)$ of G_2. Assume $H \neq G_2$. Then, by Lemma 7.10, there exists a proper open subgroup U of G_2 that contains H. But U and G_2 are now two different open subgroups of G_2 whose inverse image under μ is equal to G_1; a contradiction. □

Let $\mathcal{X}$ be a connected algebraic stack that has a moduli space (Definition 7.1). Fix a geometric point x for $\mathcal{X}$. For any other geometric point x', and any choice of a path from x' to x, we may identify the image of $\omega_{x'}$ in $\pi_1(\mathcal{X}, x')$ with a subgroup of $\pi_1(\mathcal{X}, x)$. Let $N \subseteq \pi_1(\mathcal{X}, x)$ denote the closure of the subgroup generated by all these subgroups (for various x' and various paths connecting x' to x). Then, N is a normal subgroup of $\pi_1(\mathcal{X}, x)$ that maps to zero via the homomorphism $\pi_1(\mathcal{X}, x) \to \pi_1(X_{mod}, x)$.

Theorem 7.12. *The natural map $\pi_1(\mathcal{X}, x)/N \to \pi_1(X_{mod}, x)$ is an isomorphism.*

Proof. We may assume that $\mathcal{X}$ is connected. We want to show that the map $\mu\colon \pi_1(\mathcal{X}, x) \to \pi_1(X_{mod}, x)$ induces an isomorphism

$$\bar{\mu}\colon \pi_1(\mathcal{X}, x)/N \xrightarrow{\sim} \pi_1(X_{mod}, x).$$

By Lemma 7.11, it is enough to show that μ^{-1} induces a one to one correspondence between open subgroups of $\pi_1(X_{mod}, x)$ and those open subgroups U of $\pi_1(\mathcal{X}, x)$ that contain N. If we unravel the definition of N, we see that for U to contain N is equivalent to the following: For any geometric point x' and any path γ from x to x', the image of U under the isomorphism $\pi_1(\mathcal{X}, x) \xrightarrow{\sim} \pi_1(\mathcal{X}, x')$ induced by γ contains $\pi_1^h(\mathcal{X}, x')$. By Lemma 5.1, this is equivalent to $\mathcal{X}_U \to \mathcal{X}$, the corresponding (pointed) covering space, being FPR. This shows that there is a one-to-one correspondence between open subgroups of $\pi_1(\mathcal{X}, x)$ that contain N and (pointed) connected FPR covering spaces of $\mathcal{X}$. By Lemma 7.8, the latter set is, via base change along $\pi_{mod}\colon \mathcal{X} \to X_{mod}$, in one-to-one correspondence with (pointed) connected covering spaces of (X_{mod}, x). Finally, this last set is in one-to-one correspondence with open subgroups of $\pi_1(X_{mod}, x)$. Proof is now complete. □

Remark 7.13. Proposition 7.8 remains valid if we consider étale covers that are not necessarily finite. In other words, we have an equivalence of categories between the category of fixed point reflecting étale maps to $\mathcal{X}$ and the category of étale maps to X_{mod} (if it exists). It is noteworthy that the former category always exists, regardless of existence of the course moduli space X_{mod}. What this means is that, an arbitrary algebraic stack always has a "course moduli topos" associated with it. In particular, we can talk about pro-homotopy invariants of the course moduli space of $\mathcal{X}$, even if it does not exist!

8. Application

In this section we apply the formula of Theorem 7.12 to calculate the fundamental group of the G.I.T. quotient of a groupoid actions. We will make use of the notion of a *fibration* as sketched in the appendix. All schemes (algebraic spaces, stacks) and all morphisms are assumed to be quasi-compact. **Notation:** When $R \rightrightarrows X$ is a groupoid, the notation X/R refers to the quotient as an *algebraic space* (if it exists), as opposed to $[X/R]$, which denotes the quotient as a stack. More precisely, X/R denotes the *moduli space* of $[X/R]$ (in the sense of Section 7), if it exists.

The morphisms that we call *fibrations* have the following properties (see the appendix for details):

- ▹ Let $(f,\phi)\colon (\mathcal{Y},y) \to (\mathcal{X},x)$ be a pointed fibration. Then we have an exact sequence $\pi_1(\mathcal{Y}_x,y) \to \pi_1(\mathcal{Y},y) \to \pi_1(\mathcal{X},x) \to \pi_0(\mathcal{Y}_x,y) \to \pi_0(\mathcal{Y},y) \to \pi_0(\mathcal{X},x) \to \{*\}$.
- ▹ Composition of fibrations is again a fibration. An arbitrary base extension of a fibration is a fibration.
- ▹ Being a fibration is local (on the target) in the fppf topology.
- ▹ A morphism f is a fibration if and only if f_{red} is a fibration.

We just mention a few examples of fibrations:

i) Proper flat morphisms that have geometrically reduced fibers (in particular, finite étale morphisms).
ii) Any morphism whose target is the spectrum of a field (or the spectrum of an Artin ring), and any base extension of such a morphism.
iii) The structure map of a smooth group scheme whose geometric fibers have a fixed number of connected components.
iv) In a geometric context (e.g., when everything is over $\mathbb{C}$), a morphism f that is a (quasi-)fibration in the topological sense is a fibration in our sense.

Let X be a connected scheme (or algebraic space) and let $R \overset{s,t}{\rightrightarrows} X$ be a groupoid which is flat and of finite presentation and whose diagonal map $R \to X \times X$ is separated (so the quotient stack becomes algebraic). Assume further that s (hence t) is a fibration. Fix a geometric point $x\colon \operatorname{Spec} k \to X$, and let S_x be the stabilizer group at x. Let R_x be the fiber of $s\colon R \to X$ over x. There is a natural base point for R_x, namely the image of x under the identity section $X \to R$. It follow from the definition of a groupoid, that the group of k-points of the satbilizer group scheme at x (which, by abuse of notation, we call S_x) acts on the set of k-points of R_x, and, a fortiori, on the set $\mathbf{R}_x$ of connected components of R_x. Let $x'\colon \operatorname{Spec} k' \to X$ be another geometric point. Any choice of a "path" from x' to x induces an action of $S_{x'}$ on $\mathbf{R}_x$ (because, since $R \to X$ is a fibration, $\mathbf{R}_x$ form a $\Pi_1(X)$-set). Therefore, we have an action of $S_{x'}$ on $\mathbf{R}_x$ that is well-defined up to conjugation by the action of $\pi_1(X,x)$ on $\mathbf{R}_x$. Let M_x be the largest quotient of $\mathbf{R}_x$ on which every S_x acts trivially (in other words, the collection of M_x forms a $\Pi_1(X)$-set that is the largest quotient of the $\Pi_1(X)$-set $\{\mathbf{R}_x\}$ on which the action of all S_x's is trivial). There is an alternative way of constructing M_x: Let $\mathcal{X} = [X/R]$, and consider the fibration $p\colon X \to \mathcal{X}$ (See Theorem A.12). Let

$$\pi_1(X,x) \to \pi_1(\mathcal{X},p(x)) \to \pi_0(X_{p(x)},x) \to \{*\}$$

be the corresponding fiber homotopy exact sequence. The (pointed) set $\pi_0(X_{p(x)}, x)$ is canonically isomorphic to the (pointed) set of connected components of R_x. Let $N \subseteq \pi_1(\mathfrak{X}, p(x))$ be as in Theorem 7.12. If we now kill the action of N on $\pi_0(X_{p(x)}, x)$, what we obtain is canonically isomorphic (as a (pointed) set with a $\pi_1(X, x)$ action) to M_x. Theorem 7.12 implies the following

Theorem 8.1. *Let $R \rightrightarrows X$ be a flat of finite presentation groupoid such that the diagonal $R \to X \times X$ is separated (R and X algebraic spaces) and let R/X be a quotient (i.e., a moduli space in the sense of Section 7 for the algebraic stack $[X/R]$), if it exists. Let $q\colon X \to X/R$ be the quotient map. Then we have a canonical exact sequence*

$$\pi_1(X, x) \to \pi_1(X/R, q(x)) \to M_x \to \{*\}.$$

Now, let $f\colon X \to S$ be a connected algebraic space over a connected base S, and let G be a flat, separated and of finite presentation group space over S acting on X. Assume further that $G \to S$ is a fibration. Suppose a quotient X/G exists (in the sense of the above theorem). Fix a geometric point $x\colon \operatorname{Spec} k \to X$. Let $s = f(x)$. Let $\mathbf{G}$ denote the group of connected components of G_s. Define $\mathbf{H}_x \subset \mathbf{G}$ to be the set of all connected components of G_s that contain at least one k-point that leaves x fixed. For any other choice of a geometric point x' of X, and for any choice of a path from x' to x, we can identify $\mathbf{H}_{x'}$ with a subgroup of $\mathbf{G}$. Let $\mathbf{I} \subset \mathbf{G}$ be the subgroup generated by all these groups. In this situation, Theorem 8.1 can be translated as follows

Theorem 8.2. *Let G be a group space acting on a connected algebraic spaces X, relative to a connected base S. Suppose $G \to S$ is a fibration that is flat, separated and of finite presentation. Assume a quotient X/S exists (in the sense of Theorem 8.1). Then, we have an exact sequence*

$$\pi_1(X) \to \pi_1(X/G) \to \mathbf{G}/\mathbf{I} \to \{*\}.$$

Here, $\mathbf{G}/\mathbf{I}$ is viewed as a pointed set.

In the above situation, if $X \to S$ has connected set-theoretic (as opposed to geometric) fibers, then $\mathbf{I} \subset \mathbf{G}$ could also be defined as follows: Let $\mathbf{H} \subset \mathbf{G}$ be the set of all connected components of G_s that contain at least one k-point whose action on X_s has a fixed point. For any other choice of a geometric point s' that lies in the image of X, and for any choice of a path from s' to s coming from X, we can identify $\mathbf{H}'$ (the counterpart of $\mathbf{H}$ at s') with a subgroup of $\mathbf{G}$. Then $\mathbf{I} \subset \mathbf{G}$ is the subgroup generated by all these groups. Also, the same description for $\mathbf{I}$ is valid when the action of $\pi_1(X)$ (via that of $\pi_1(S)$) on the set of connected components of the geometric fibers of $G \to S$ is trivial. In this case, $\mathbf{I} \subset \mathbf{G}$ will indeed be a normal subgroup and the exact sequence of Theorem 8.2 will become an exact sequence of groups. The following corollaries are instances where this phenomenon occurs.

Corollary 8.3. *Let X be a connected algebraic space, and let G be a finite group acting on X (relative to a certain base S). Let X/G be the quotient for this action. Let $I \subset G$ be the subgroup generated by all elements that have fixed points. Then, I is normal in G, and we have the following exact sequence of groups:*

$$\pi_1(X) \to \pi_1(X/G) \to G/I \to 1.$$

Corollary 8.4. *Let X be a connected algebraic space over an algebraically closed field k, and let G be an algebraic group (not necessarily reduced) over k acting on*

X. Assume a quotient X/G exists (in the sense of Theorem 8.1). Let $\mathbf{G}$ be the group of connected components of G, and let $\mathbf{I} \subset \mathbf{G}$ be the set of all components that contain at least one element that leaves some point of X fixed. Then, $\mathbf{I}$ is a normal subgroup of $\mathbf{G}$, and we have the following exact sequence of groups:

$$\pi_1(X) \to \pi_1(X/G) \to \mathbf{G}/\mathbf{I} \to 1.$$

Corollary 8.5. *Let X and G be as in Theorem 8.2. Assume further that X is simply connected. Then, we have an isomorphism $\pi_1(X/G) \xrightarrow{\sim} \mathbf{G}/\mathbf{I}$.*

8.1. **Description of the kernel of $\pi_1(X) \to \pi_1(X/G)$.** If we go through the steps that resulted in the proof of Theorem 8.1, we will see that the kernel of $\pi_1(X) \to \pi_1(X/G)$ is, vaguely speaking, the smallest subgroup of $\pi_1(X)$ conaining all the elements that "obviously" map to zero. To illustrate this idea, let us consider the typical situation where G is a discrete group acting on a topological space.[1] Let X/G be the usual topological quotient, and let $q\colon X \to X/G$ denote the quotient map. Pick an element $g \in G$ and a fixed point $y \in X$ for g. Let $x \in Y$ be an arbitrary point and γ a path joining x to y. Then $\gamma g(\gamma^{-1})$, the juxtaposition of γ and $g(\gamma^{-1})$, is a path from x to $g(x)$, whose image in X/G is a trivial loop at $q(x)$. Let us call a path that is of the form $\gamma g(\gamma^{-1})$, or more generally any path that can be obtained by composing a finite number of such paths, a *doomed* path. In particular, we can talk about doomed loops around x, namely, doomed paths which start from x and end at x. Doomed loops form a normal subgroup of $\pi_1(X)$, and they obviously map to zero in $\pi_1(X/G)$. It follows from the proof of Theorem 8.1 that, in fact, the group of doomed loops is equal to the kernel of $\pi_1(X) \to \pi_1(X/G)$.

When G is no longer discrete, there are some more loops that "obviously" map to zero in $\pi_1(X/G)$; namely, the ones coming from the loops in G. More precisely, let x be a point in X. Then, we have a map from G to X that sends g to $g(x)$. The image of the fundamental group of (the connected component of the identity of) G in $\pi_1(X,x)$ lies in the kernel of $\pi_1(X) \to \pi_1(X/G)$. Again, it follows from the proof of Theorem 8.1 that this group, together with the group of doomed loops, generate the kernel of $\pi_1(X) \to \pi_1(X/G)$.

9. More examples

Example 9.1. *An example from number theory.* This example is to illustrate the relationship between the notion of ramification on an algebraic stack to the classical notion of ramification in algebraic number theory. Let $K \subset L$ be a Galois extension of number fields with Galois group G. Let $\mathcal{O}_K \subset \mathcal{O}_L$ be the corresponding rings of integers. Let $\mathcal{X} = [\operatorname{Spec} \mathcal{O}_L/G]$. The moduli space of $\mathcal{X}$ is $\operatorname{Spec} \mathcal{O}_L/G = \operatorname{Spec} \mathcal{O}_K$. The ramified points of $\mathcal{X}$ corresponds to primes of $\mathcal{O}_K$ that are ramified in $\mathcal{O}_L$, and the hidden fundamental group at each ramified point is isomorphic to the inertia group at the corresponding prime. Therefore, the group $I \subseteq G$ introduced in Corollary 8.3 is equal to the subgroup generated by all the inertia groups for various primes in $\mathcal{O}_K$. The exact sequence of Corollary 8.3 now takes the form

$$\pi_1(\operatorname{Spec} \mathcal{O}_L) \to \pi_1(\operatorname{Spec} \mathcal{O}_K) \to G/I \to 1,$$

[1]The results of this paper can be naturally generalized to topological/combinatorial stacks, in which case instead of the algebraic fundamental group we work with the usual fundamental group. The topological approach is the subject of a forthcoming paper.

which is the same as the exact sequence

$$\mathrm{Gal}(\mathcal{O}_L^{ur}/\mathcal{O}_L) \to \mathrm{Gal}(\mathcal{O}_K^{ur}/\mathcal{O}_K) \to G/I \to 1,$$

whose validity can be checked readily by easy number theoretic arguments. Here, $\mathcal{O}_L^{ur}$ stands for the maximal unramified extension.

Example 9.2. *Weighted projective spaces are simply connected.* Let k be an algebraically closed field. Let $n_1, n_2, \cdots, n_d$ be a sequence of integers. Consider the action of k^* on $\mathbb{A}^d \backslash \{0\}$ which, for any $\zeta \in k^*$, sends $(x_1, x_2, \cdots, x_d) \in \mathbb{A}^k \backslash \{0\}$ to $(\zeta^{n_1} x_1, \zeta^{n_2} x_2, \cdots, \zeta^{n_d} x_d)$. Let $\mathbb{P}(n_1, n_2, \cdots, n_d) := \mathbb{A}^d \backslash \{0\}/k^*$ be the quotient of this action. It is an algebraic variety which is called the *weighted projective space* of weight $(n_1, n_2, \cdots, n_d)$. It is the moduli space of $\mathcal{P}(n_1, n_2, \cdots, n_d) := [\mathbb{A}^d \backslash \{0\}/k^*]$. The quotient map $\mathbb{A}^d \backslash \{0\} \to \mathcal{P}(n_1, n_2, \cdots, n_d)$ is a fibration. An easy fiber homotopy exacts sequence argument shows that $\mathcal{P}(n_1, n_2, \cdots, n_d)$ is simply connected. Therefore, $\mathbb{P}(n_1, n_2, \cdots, n_d)$ is also simply connected by Theorem 7.12.

Example 9.3. $\bar{\mathcal{M}}_{1,1}$. It is well-known that the compactified moduli stack $\bar{\mathcal{M}}_{1,1}$ of elliptic curves (over, say, $\mathbb{C}$) is isomorphic to $\mathbb{P}(4,6)$, which is simply connected. Its moduli space is the j-line, which is simply connected, as anticipated by Theorem 7.12.

Example 9.4. $\mathcal{M}_{1,1}$. The non-compact moduli stack of elliptic curves is the quotient of the upper half-plane by the action of $SL_2(\mathbb{Z})$. Therefore, its (topological) fundamental group is isomorphic to $SL_2(\mathbb{Z})$, and its algebraic fundamental group is isomorphic to $\widehat{SL_2(\mathbb{Z})}$, the profinite completion of $SL_2(\mathbb{Z})$. The standard generators S and T together with $-I$ generate $SL_2(\mathbb{Z})$, and they all have fixed points. Therefore, by Corollary 8.5, the quotient of the upper half-plane by the action of $SL_2(\mathbb{Z})$ (i.e., the moduli space of $\mathcal{M}_{1,1}$) should be simply connected, which is obviously the case, since it is isomorphic to $\mathbb{C}$. (To be more precise, in order to be able to make use of Corollary 8.5, we should first take a normal subgroup of $SL_2(\mathbb{Z})$ that acts fixed-point-freely on the upper half-plane, and then apply Corollary 8.5 to the action of the cokernel of this group on the quotient space, which is now an algebraic curve. Otherwise, we need to use the topological counterpart of Corollary 8.5, which is indeed true.)

Next we show that the stack of Example 5.11 is simply-connected. For this, we use the following proposition, the proof of which uses results from the Part II of the paper.

Proposition 9.5. *Let $\mathcal{X}$ be a connected Deligne-Mumford gerbe with finite (étale) stabilizer. Then, for every geometric point x of $\mathcal{X}$, the sequence*

$$\pi_1^h(\mathcal{X}, x) \to \pi_1(\mathcal{X}, x) \to \pi_1(X_{mod}, x) \to 1$$

is exact. If this sequence is exact on the left for some x, then $\mathcal{X}$ is uniformizable.

Proof. The exact sequence follows from Corollary 10.5. The last part follows from Proposition 10.4 and Theorem 6.2. □

Example 9.6. (*Example 5.11 revisited.*) We have $\pi_1^h(\mathcal{X}, x) \cong \mathbb{Z}/2\mathbb{Z}$ for all points $x \in \mathcal{X}$. Since $\mathbb{P}^1$ is simply connected, $\mathcal{X}$ does not admit any non-trivial finite étale cover by a scheme, nor even by an algebraic space ([7], Corollary 6.16), because, otherwise, composing it with π_{mod} would give us a non trivial finite étale cover of $\mathbb{P}^1$ (note that the moduli map $\mathcal{X} \to \mathbb{P}^1$ is finite étale). Therefore, $\mathcal{X}$ is not uniformizable. So by Theorem 9.5, all the maps $\omega_x\colon \pi_1^h(\mathcal{X},x) \to \pi_1(\mathcal{X},x)$ are zero maps and $\pi_1(\mathcal{X},x)$ is trivial. In other words, $\mathcal{X}$ does not admit any non-trivial finite étale cover.

The next pathological example is a good exercise for playing around with the ideas presented in this paper. It was suggested to me by J. de Jong.

Example 9.7. (*A non simply-connected, non-uniformizable stack.*) Let $N \subset \mathbb{P}^3_{\mathbb{C}}$ be the hypersurface defined by the homogeneous equation $X^4 + Y^4 + Z^4 + W^4 = 0$. Let ζ be a primitive fourth root of unity. The action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{P}^3_{\mathbb{C}}$ given by

$$(x : y : z : w) \mapsto (x : \zeta y : \zeta^2 z : \zeta^3 w)$$

induces a fixed point free action on N. Let M be the quotient of this action. The fundamental group of the scheme M is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Let $I = \mathbb{Z}/3\mathbb{Z}$. From the spectral sequence

$$H^q(\pi_1(M), H^p(N,I)) \Rightarrow H^{p+q}(M,I)$$

we obtain the following exact sequence:

$$H^2(\pi_1(M),I) \hookrightarrow H^2_{et}(M,I) \to H^2_{et}(N,I)^{\pi_1(M)} \to H^3(\pi_1(M),I).$$

Note that the action of $\pi_1(M)$ on I is trivial. So, we have $H^2_{et}(N,I)^{\pi_1(M)} = H^2_{et}(N,I)$. On the other hand, $H^3(\pi_1(M),I) = H^3(\mathbb{Z}/4\mathbb{Z},\mathbb{Z}/3\mathbb{Z}) = 0$. Therefore, the above exact sequence looks as follows:

$$0 \to H^2(\pi_1(M),I) \to H^2_{et}(M,I) \to H^2_{et}(N,I) \to 0.$$

Since N is projective, $H^2_{et}(N,I)$ is non-trivial. In particular, $H^2_{et}(M,I)$ is strictly bigger than $H^2(\pi_1(M),I)$. It is shown in ([11], Section 6) that an element in $H^2_{et}(M,I)$ that is not in the image of $H^2(\pi_1(M),I)$ corresponds to a gerbe $\mathcal{X}$ over M that is *non-uniformizable*. The automorphism group of this gerbe is $\mathbb{Z}/3\mathbb{Z}$. By Theorem 9.5, we have the following exact sequence that is *not* left exact:

$$\mathbb{Z}/3\mathbb{Z} \to \pi_1(\mathcal{X}) \to \mathbb{Z}/4\mathbb{Z} \to 0.$$

Therefore, $\pi_1^h(\mathcal{X}) \to \pi_1(\mathcal{X})$ is the zero map, and $\pi_1(\mathcal{X}) \xrightarrow{\sim} \pi_1(X_{mod}) = \mathbb{Z}/4\mathbb{Z}$ by Theorem 7.12 (or Theorem 9.5 above).

Part 2. Further results

Part two of this paper is in a way a technical companion to Part one. The first main result in this part (Theorem 11.4), which is a generalization of Theorem 6.2 of Part one, tells us to what extent one can resolve stackyness after by passing to covering spaces. The second main result is a finiteness theorem about the kernel of the homomorphism $\pi_1(\mathcal{X},x) \to \pi_1(X_{mod},x)$ (Theorem 12.1). The main tool in proving these result is the Stratification Theorem 11.3, which we believe is useful in its own right.

Let us first reformulate the results of Section 5 in a form that is more suitable for our next applications:

Proposition 9.8. *Let $(\mathcal{X},x)$ be a connected algebraic stack. Then, the (isomorphism classes of) pointed covering spaces $(\mathcal{Y},y)$ of $(\mathcal{X},x)$ that are FPR at y, are in one-to-one correspondence with open-closed subgroups of $\pi_1(\mathcal{X},x)$ that contain the image of $\omega_x \colon \pi_1^h(\mathcal{X},x) \to \pi_1(\mathcal{X},x)$. The (isomorphism classes of) pointed covering spaces $(\mathcal{Y},y)$ of $(\mathcal{X},x)$ that are FPR at every y' in the fiber of x, are in one-to-one correspondence with open-closed subgroups of $\pi_1(\mathcal{X},x)$ that contain the normal closure of the image of $\omega_x \colon \pi_1^h(\mathcal{X},x) \to \pi_1(\mathcal{X},x)$.*

If x' is another point in $\mathcal{X}$ (possibly identical to x), we can give a similar description of pointed covers of $(\mathcal{X},x)$ that are FPR for all point above x'. To do so, we take the normal closure of the image of $\omega_{x'}$ in $\pi_1(\mathcal{X},x')$, and transfer it into a subgroup of $\pi_1(\mathcal{X},x)$ via a path joining x' and x. This subgroup is well defined and can be identified with $\Phi_{x'}(x)$, where $\Phi_{x'}$ is the normal subgroupoid of $\Pi_1(\mathcal{X})$ generated by the image of $\omega_{x'}$ (See Section 2.2 for definitions and notations). The pointed covering spaces $(\mathcal{Y},y)$ of $(\mathcal{X},x)$ that are FPR at every point above y' are in one-to-one correspondence with open-closed subgroups of $\pi_1(\mathcal{X},x)$ that contain $\Phi_{x'}(x)$. Indeed, the same thing is true if we replace x' with a set S of points of $\mathcal{X}$:

Proposition 9.9. *Let $(\mathcal{X},x)$ be a connected pointed stack. Let S be a set of geometric points of $\mathcal{X}$. Let $\Phi_S \subseteq \Pi_1(\mathcal{X})$ be the normal closure of the subgroupoid generated by images of ω_x for $x \in S$. Then, the (isomorphism classes) of pointed covering spaces of $\mathcal{X}$ that are FPR at all geometric points y of $\mathcal{Y}$ lying above points in S, are in one-to-one correspondence with open-closed subgroups of $\pi_1(\mathcal{X},x)$ that contain $\Phi_S(x)$.*

This proposition will be used later in the proof of Theorem 12.1.

10. Some Galois theory of gerbes

In this section we introduce a certain class of gerbes, called *monotonous gerbes* (Definition 10.1), and study their Galois theory. The point is that the maps $\omega_x \colon \pi_1^h(\mathcal{X},x) \to \pi_1(\mathcal{X},x)$ behave nicely for monotonous gerbes. We will see in the next section that every algebraic stack has a stratification by monotonous gerbes.

We recall a fact from ([3], Exposé VIB): Let $G \to X$ be a flat group space of finite type. Let G^o be the functor of *connected component of the identity*; that is, the functor that associates to any X-scheme T, the set of all T points $g \in G(T)$ of G for which $g(t)$ falls within the connected component of identity of G_t for every

$t \in T$. This functor is a subsheaf of G. If this functor is representable, then G^o is an open subgroup space of G.

Definition 10.1. We say that a finite type flat group space $G \to X$ is *monotonous*, if the number of geometric connected components of the fiber of point $x \in X$ is a locally constant function on X, and if the functor G^o defined above is representable (necessarily by an open subspace of G). We say that an algebraic stack $\mathcal{X}$ is a *monotonous gerbe*, if it is connected and its stabilizer group is monotonous.

The monotonicity is a rough approximation of being a fibration. For monotonous gerbes the stabilizer groups and hidden fundamental groups vary in a more tractable way, as we will see in the following results (see also Remark 10.6). Every Deligne-Mumford gerbe with finite stabilizer is automatically monotonous. Of course, this is not true if the stabilizer is not finite (Example 5.9).

It follows from ([9], Corollaire 10.8), that a monotonous gerbe is indeed an fppf gerbe, that it has a "moduli space" X_{mod}, and that the moduli map $\mathcal{X} \to X_{mod}$ is smooth and of finite type. In fact, it is easy to see that this "moduli space" is a moduli space in the sense of Section7, but we will not need this fact here.

If G is a monotonous group space over S, then G/G^o is a group space that is unramified and of constant degree over S. Therefore, it is finite étale over S. The following lemma is now immediate.

Lemma 10.2. *Let X be a scheme, and let $G \to X$ be a monotonous group space. Let H be a subgroup that is both closed and open. Then, H is monotonous.*

Proposition 10.3. *Let $\mathcal{X}$ be a connected monotonous gerbe, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be a connected covering space. Then:*

- **i)** *$\mathcal{Y}$ is a monotonous gerbe.*
- **ii)** *If f is FPR (Definition 7.5) at some geometric point of $\mathcal{Y}$, then it is FPR everywhere.*

Proof. Let $p\colon X \to \mathcal{X}$ be a chart for $\mathcal{X}$ and let $q\colon Y \to \mathcal{Y}$ be a connected component of the chart for $\mathcal{Y}$ obtained by pulling back X via f. By Corollary 2.3, the natural homomorphism $S_Y \to S_X \times_X Y$ is an open closed immersion. So it follows from Lemma 10.2, $S_Y \to Y$ is monotonous. This proves (**i**). Since $S_Y \to S_X \times_X Y$ is an open closed immersion of monotonous spaces, the necessary and sufficient condition for it to be an isomorphism is that the number of the geometric components of the fibers of S_Y and $S_X \times_X Y$ be equal. But this can be checked at a single point. This proves (**ii**). □

Proposition 10.4. *Let $\mathcal{X}$ be a monotonous gerbe. Then, the image of $\Omega_{\mathcal{X}}\colon \Pi_1^h(\mathcal{X}) \to \Pi_1(\mathcal{X})$ is a normal subgroupoid. In particular, the image of ω_x is a normal subgroup of $\pi_1(\mathcal{X}, x)$ for every geometric point x, and all these images are (non canonically) isomorphic.*

Proof. Fix a point x in $\mathcal{X}$. Let $I \subseteq \Pi_1(\mathcal{X})$ be the image of $\Omega_{\mathcal{X}}$, and let $N \subseteq \Pi_1(\mathcal{X})$ be its normal closure. Using Proposition 9.9, we can translate Proposition 10.3 as saying that, an open-closed subgroup of $\pi_1(\mathcal{X}, x)$ contains $N(x)$ if and only if it contains its subgroup $I(x)$. This implies that the closure of $N(x)$ and $I(x)$ are equal. But $I(x)$, being finite, is already closed. So $I(x) = N(x)$, which proves the claim. □

Corollary 10.5. *Let $(\mathcal{X}, x)$ be a monotonous gerbe. Then we have the following exact sequence:*

$$\pi_1^h(\mathcal{X}, x) \to \pi_1(\mathcal{X}, x) \to \pi_1(X_{mod}, x) \to 1.$$

Proof. This follows from Proposition 10.4 and Theorem 7.12. □

Remark 10.6. The moduli map $\pi_{\mathcal{X}} \colon \mathcal{X} \to X_{mod}$ of a monotonous gerbe resembles a "fibration with connected fibers", and the above exact sequence is the analogue of the "fiber homotopy exact sequence".

Remark 10.7. The monotonous stacks for which the above sequence is short exact have quasi-finite stabilizers (Example 4.3). If we restrict ourselves to reduced Deligne-Mumford stacks, then there is a simple description of the category of such stacks ([11], Section 6) which says that a stack with this property is uniquely determined by its moduli space, together with the above short exact sequence. In other words, to give a reduced monotonous Deligne-Mumford gerbe is the same as to give a reduced algebraic space plus an extension of its fundamental group by a finite group. To reconstruct the gerbe out of this data, the idea is to take the "universal cover $\widetilde{X_{mod}}$" and divide it by the middle term of the group extension sequence (acting via the quotient homomorphism onto $\pi_1(X_{mod})$). We will not be using this result in this paper, so we won't go into more details. It is worthwhile to mention though that, using this result, one obtains a simple description of the category of zero-dimensional reduced Deligne-Mumford stacks (*loc. cit.*, Section 6.1), as any such stack is monotonous.

Corollary 10.8. *Let $\mathcal{X}$ be an algebraic stack, and let $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ be a locally closed immersion (or, simply, a monomorphism) of stacks. Assume $\mathcal{X}_0$ is monotonous. Let x and x' be geometric points of $\mathcal{X}$ that factor through $\mathcal{X}_0$. Then, the image of $\omega_x \colon \pi_1^h(\mathcal{X}, x) \to \pi_1(\mathcal{X}, x)$ is isomorphic (non canonically) to the image of $\omega_{x'} \colon \pi_1^h(\mathcal{X}, x') \to \pi_1(\mathcal{X}, x')$. In particular, if ω_x is the zero map, then so is every other $\omega_{x'}$. Finally, if ω_x is injective, then so is $\omega_{x'}$.*

Proof. Let $I_0 \subseteq \pi_1(\mathcal{X}_0, x)$ be the image of $\pi_1^h(\mathcal{X}_0, x) \to \pi_1(\mathcal{X}_0, x)$, and $I \subseteq \pi_1(\mathcal{X}, x)$ be the image of $\pi_1^h(\mathcal{X}, x) \to \pi_1(\mathcal{X}, x)$. Define $I_0' \subseteq \pi_1(\mathcal{X}_0, x')$ and $I' \subseteq \pi_1(\mathcal{X}, x')$ similarly. Note that $\pi_1^h(\mathcal{X}_0, x)$ is canonically isomorphic to $\pi_1^h(\mathcal{X}, x)$ (Proposition 3.2). So I is the image of I_0 under the the natural map $\pi_1(\mathcal{X}_0, x) \to \pi_1(\mathcal{X}, x)$ (similarly for I'). Fix a "path" from to x to x' in $\mathcal{X}_0$. We saw in the proof of the previous proposition, that this "path" induces an isomorphism $\pi_1(\mathcal{X}_0, x) \xrightarrow{\sim} \pi_1(\mathcal{X}_0, x')$, that maps I_0 isomorphically to I_0'. The image of this "path" in $\mathcal{X}$, induces an isomorphism $\pi_1(\mathcal{X}, x) \xrightarrow{\sim} \pi_1(\mathcal{X}, x')$, that, because of the commutativity of the following diagram, maps I isomorphically onto I':

$$\begin{array}{ccc} \pi_1(\mathcal{X}_0, x) & \longrightarrow & \pi_1(\mathcal{X}, x) \\ \downarrow \sim & & \downarrow \sim \\ \pi_1(\mathcal{X}_0, x') & \longrightarrow & \pi_1(\mathcal{X}, x') \end{array}$$

To prove the final statement, note that, if $\pi_1^h(\mathcal{X}, x)$ is injective, then the stabilizer group of x is zero dimensional (Example 4.3) and its set of geometric connected components, which is a finite set, is isomorphic to $\pi_1^h(\mathcal{X}, x)$. Therefore, since the stabilizer group $\mathcal{S}_{\mathcal{X}_0} \to \mathcal{X}_0$ is flat, its relative dimension should be zero. In particular, the stabilizer of x' is also zero dimensional and its set of geometric connected components is isomorphic to $\pi_1^h(\mathcal{X}, x')$. Because $\mathcal{S}_{\mathcal{X}_0} \to \mathcal{X}_0$ is monotonous, $\pi_1^h(\mathcal{X}, x)$ and $\pi_1^h(\mathcal{X}, x')$ have the same number of elements. On the other hand, we just proved that their images under, respectively ω_x and $\omega_{x'}$, have the same number of elements. Therefore, if ω_x is injective, so will be $\omega_{x'}$. □

11. Stratification by gerbes and its applications

The stratification theorem of this section is just a technical modification of that of [9], and so is its proof. We prove that every algebraic stack has an stratification by monotonous gerbes. We then use this stratification to prove a stronger version of Theorem 6.2.

We will need the following lemma, due to Kai Behrend ([1], Section 5), in the proof of our Stratification Theorem (Section 11).

Lemma 11.1. *Let $G \to X$ be a finite type group space, and assume X is a Noetherian scheme. Then, X can be written as a disjoint union of a finite family $(X_i)_{i \in I}$ of reduced locally closed subschemes, such that for every $i \in I$, the restriction $f_i \colon G \times_X X_i \to X_i$ is a group scheme for which the functor of* connected componets of the identity *(see the beginning of Section 10) is representable.*

Lemma 11.2. *Let $f \colon Y \to X$ be a morphism of finite presentation between Noetherian schemes. Then, X can be written as a disjoint union of a finite family $(X_i)_{i \in I}$ of reduced locally closed subschemes such that for every $i \in I$, the restriction $f_i \colon Y \times_X X_i \to X_i$ is flat and has constant number of geometric fibers.*

Proof. Lemma follows from the theorem on generic flatness ([5], Théorème 6.9.1), and generic constancy of the number of geometric connected components ([5], Proposition 9.7.8). □

Theorem 11.3. *Let $\mathcal{X}$ be a Noetherian algebraic stack. Then, $\mathcal{X}$ can be written as a disjoint union of a finite family $(\mathcal{X}_i)_{i \in I}$ of locally closed subsets such that each $\mathcal{X}_i$ with its reduced structure is a monotonous gerbe.*

Proof. In the proof given in ([9], Théorème 11.5), simply substitute Lemma 11.2 and Lemma 11.1 above for the 'generic flatness theorem', and proof goes through verbatim. □

Now we can prove the stronger version of Theorem 6.2.

Theorem 11.4. *Let $\mathcal{X}$ be a Noetherian algebraic stack. Then, there exists a covering space $\mathcal{Y} \to \mathcal{X}$ such that ω_y is the zero map for all geometric points y of $\mathcal{Y}$.*

Proof. We may assume $\mathcal{X}$ is connected. Let $(\mathcal{X}_i)_{i \in I}$ be the stratification of $\mathcal{X}$ by monotonous gerbes (Theorem 11.3). Let S be a finite subset of $|\mathcal{X}|$ that intersects all $\mathcal{X}_i$. Fix a base point x for $\mathcal{X}$. For any $x' \in S$, choose a path connecting x' to x, and use that to transfer the image of $\omega_{x'}$ onto a subgroup of $\pi_1(\mathcal{X}, x)$. The union of all these subgroups is a finite subset of $\pi_1(\mathcal{X}, x)$ (see Example 4.3), so we could choose an open normal subgroup of finite index $U \subseteq \pi_1(\mathcal{X}, x)$ that does not intersect any of these subgroups. The corresponding covering space will have the desired property (Lemma 5.1 and Corollary 10.8). □

Let S be a subset of $|\mathcal{X}|$ that intersects all the $\mathcal{X}_i$. Assume ω_x is the zero map for all x in S. As in the proof of the above theorem, Lemma 5.1 together with Corollary 10.8 imply that ω_x is the zero map for all x in $\mathcal{X}$. In particular, let $S \subset |\mathcal{X}|$ be a set of points in $\mathcal{X}$ that has the following property: For every locally closed subset T of $|\mathcal{X}|$, the intersection $T \cap S$ is nonempty. Then, once the injectivity of ω_x is established for all $x \in S$, then it follows that ω_x is injective for all x. For instance, the set of closed point of an algebraic stack of finite type over a field has this property.

Let $\mathcal{Y} \to \mathcal{X}$ be a covering space of $\mathcal{X}$. Let y be a point in $\mathcal{Y}$ and $x = f(y)$ its image in $\mathcal{X}$. Theorem 11.4 is most useful when combined with the following facts (that are always true):

- The kernel of ω_y is the same as the kernel of ω_x.
- The stabilizer of $\mathcal{Y}$ is an open closed subgroup of the pull back of the stabilizer of $\mathcal{X}$.

For example, if in Theorem 11.4 we assume in addition that ω_x is injective for every geometric point x of $\mathcal{X}$, then it follows that the hidden fundamental groups of $\mathcal{Y}$ are all trivial. In other words, the stabilizer group of $\mathcal{Y}$ has zero-dimensional geometrically connected fibers. If we further assume that $\mathcal{X}$ has unramified stabilizer (i.e., $\mathcal{X}$ is Deligne-Mumford), then so does $\mathcal{Y}$. Therefore, the stabilizer of $\mathcal{Y}$ will be trivial, which means that $\mathcal{Y}$ is an algebraic space (compare Theorem 6.2).

12. A finiteness theorem

All the algebraic stack in this section are assumed to be Noetherian. We prove the following

Theorem 12.1. *Let $(\mathcal{X}, x)$ be a connected algebraic stack that has a moduli space (in the sense of Section 7). Let N be the kernel of the $\pi_1(\mathcal{X}, x) \to \pi_1(X_{mod}, x)$. Then, there exists a finite set T of torsion elements in N such that N is the smallest closed normal subgroup containing T.*

The main ingredient of the proof is the following result, which can be thought of as a generalization of Proposition 10.3

Proposition 12.2. *Let $\mathcal{X}$ be and algebraic stack Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a Galois covering space. Let $(\mathcal{X}_i)_{i\in I}$ be a stratification of $\mathcal{X}$ by monotonous substacks (Theorem 11.3). Let S be a subset of $\mathcal{X}$ that intersects all the $\mathcal{X}_i$. Assume the maps $\pi_1^h(\mathcal{Y}, y) \to \pi_1^h(\mathcal{X}, x)$ are isomorphisms for all geometric points y of $\mathcal{Y}$ lying above points in S. Then, f is fixed point reflecting.*

Proof. Let $\mathcal{Y}_i = \mathcal{X}_i \times_{\mathcal{X}} \mathcal{Y}$. Clearly $\mathcal{Y}$ is a disjoint union of $\mathcal{Y}_i$. Since the hidden fundamental group of a point only depends on its residue gerbe, it is enough to prove that each $\mathcal{Y}_i \to \mathcal{X}_i$ is fixed point reflecting. Each $(\mathcal{Y}_i)$ is a disjoint union of connected algebraic stacks that are all isomorphic to each other (because the action of the Galois group permutes them around transitively). So it is enough to prove the proposition for one of these connected components. Such a connected component is a monotonous gerbe by Proposition 10.3 (**i**), so the result follows from Proposition 10.3 (**ii**) □

Proof of Theorem 12.1. Fix a finite set S as in Proposition 12.2. In this proof, we will switch our notation for elements of S from x to s. Fix a geometric point x for $\mathcal{X}$ all through the proof. Let $\Phi \subseteq \Pi_1(\mathcal{X})$ be the closed normal subgroupoid of $\Pi_1(\mathcal{X})$

generated by the images of ω_x for various $s \in S$, and let $N = \Phi(x)$. The group N has the desired finiteness property of the theorem. We claim that $\pi_1(\mathcal{X}, x)/N$ is naturally isomorphic to $\pi_1(X_{mod}, x)$. The set of open-closed normal subgroups of $\pi_1(X_{mod}, x)$ is in one-to-one correspondence with the set of Galois covering spaces of X_{mod}. By Proposition 7.8, the set of Galois covering spaces of X_{mod} is in one-to-one correspondence with the set of fixed point reflecting Galois covering spaces of $\mathcal{X}$. By Proposition 12.2 and Proposition 9.9, this set is in one-to-one correspondence with the set of open-closed normal subgroupoids of $\Pi_1(\mathcal{X})$ that contain the image of ω_x for all $s \in S$ (equivalently, contain Φ). This set is in one-to-one correspondence with open-closed normal subgroups of $\pi_1(\mathcal{X}, x)$ containing N. Claim now follows from Lemma 7.11. $\square$

Appendix: Fibrations

In this section we introduce a notion of a *fibration* between algebraic spaces and prove that being a fibration is local in the fppf topology. The results of this appendix are used in section 8. This appendix is divided into four subsections. In the first part, we define fibrations in terms of the fiber homotopy exact sequence of fundamental groups. In the second part, we discuss the geometric meaning of being a fibration. In the third part, we prove that fibrations satisfy fppf descent. In the last part we discuss fibrations of algebraic stacks.

In this appendix all algebraic spaces and morphisms between them are assumed to be quasi-compact.

A.1. **Fibrations: the definition.** Let $f\colon Y \to X$ be a morphism of algebraic spaces. We would like to study those f for which the following axiom holds:

A) For any choice of base point y and $x = f(y)$, the sequence

$$\pi_1(Y_x, y) \to \pi_1(Y, y) \to \pi_1(X, x) \to \pi_0(Y_x, y) \to \pi_0(Y, y) \to \pi_0(X, x) \to \{*\} \tag{2}$$

is exact.

When X and Y are connected, the above exact sequence takes the following form:

$$\pi_1(Y_x, y) \to \pi_1(Y, y) \to \pi_1(X, x) \to \pi_0(Y_x, y) \to \{*\} \tag{3}$$

The axiom **A** means that this shorter sequence is exact for the restriction of f to any connected component of X (and the corresponding component of Y).

Definition A.3. A surjective morphism $f\colon Y \to X$ is called a *quasi-1-fibration*, or a *fibration* for short, if every base extension of f satisfies **A**.

A well-known class of fibrations are covering (i.e., finite étale) morphisms. More generally, it is proven in ([4], Exposé X, 1.6) that any proper flat morphism $f\colon Y \to X$ that has geometrically reduced fibers is a fibration (X locally Noetherian). The following proposition is a rather tedious exercise in diagram chasing:

Proposition A.4. *Let $g\colon Z \to Y$ be a fibration, and let $f\colon Y \to X$ be morphisms that satisfies* **A**. *Then $f \circ g$ satisfies* **A**. *If f and g are both fibrations, then so is $f \circ g$.*

A.2. **Geometric interpretation.**

Definition A.5. By a *geometrically connected* morphism of algebraic spaces we mean the one whose geometric fibers are connected.

In this section we analyze the geometric meaning of **A** for geometrically connected morphisms $f\colon Y \to X$. In this case, the exact sequence (2) takes the following form

$$\pi_1(Y_x, y) \to \pi_1(Y, y) \to \pi_1(X, x) \to 1. \tag{4}$$

Let us translate this exactness into geometry. Assume $f\colon (Y, y) \to (X, x)$ is a pointed morphism (not necessarily geometrically connected) of connected algebraic spaces. Then we have the following

Proposition A.6. *Let X' be covering space of X, and let $U \subset \pi_1(X,x)$ be the corresponding subgroup (for some choice of base point). Let Y' be the pull-back of X' via f. Then, there is a natural bijection between the set of connected components of Y' and the set of orbits of the (right) action (through f) of $\pi_1(Y,y)$ on the set $U\backslash\pi_1(X,x)$ of right cosets of U. In particualr, $\pi_1(Y,y) \to \pi_1(X,x)$ is surjective if and only if the pull back via f of every (connected) covering space X' of X is connected.*

Proposition A.7. *The sequence (4) being exact is equivalent to the following:*

- *A covering space $P: Y' \to Y$ descends to a covering space of X if and only if the restriction of Y' over the geometric fiber Y_x has a section. Furthermore, the descended covering space is unique (up to isomorphism).*

It is easy to see that the uniqueness of descent in the above proposition is valid for non connected finite étale covers as well.

Let us recall a well-known

Lemma A.8. *Let $f\colon (Y,y) \to (X,x)$ be a pointed morphism of connected algebraic spaces. Let $(X',x') \to (X,x)$ be a pointed covering space. Then, f lifts (necessarily uniquely) to (X',x') if and only if the image of $\pi_1(Y,y)$ in $\pi_1(X,x)$ under f is contained in the image of $\pi_1(X',x')$.*

Proof. One implication is trivial. To prove the non trivial one, assume the image of $\pi_1(Y,y)$ in $\pi_1(X,x)$ under f is contained in the image of $\pi_1(X',x')$. Consider the pull back of (X',x') alonf f. This pull back is a disjoint union of covering spaces of (Y,y), and it is naturally pointed. Let (Y',y') be the connected component that contains the base point. It corresponds to the open subgroup of $\pi_1(Y,y)$ which, by hypothesis, is equal to the entire $\pi_1(Y,y)$. So $(Y',y') \to (Y,y)$ is trivial. The inverse map $(Y,y) \to (Y',y')$ composed with $(Y',y') \to (X',x')$ is the required lift. □

Proposition A.9. *Let $f\colon Y \to X$ be a morphism of connected algebraic spaces that satisifies* **A**. *Then, there is a factorization $f = p \circ g$ such that p is finite étale and g is geometrically connected. Furthermore, g satisfies* **A**. *The decomposition $f = p \circ g$ is unique up to a unique isomorphism and commutes with base change.*

Proof. For any choice of a base point y for Y we have the following exact sequence:

$$\pi_1(Y_x,y) \to \pi_1(Y,y) \to \pi_1(X,x) \to \pi_0(Y_x,y) \to \{*\},$$

where $x = f(y)$. Since the map $\pi_1(Y,y) \to \pi_1(X,x)$ is independent of the choice of the base-point y (up to isomorphism), the cardinality of $\pi_0(Y_x,y)$ will also be independent of y. So if we move y around, we see that, as long as x is in the image of f, the number of geometric connected components of Y_x remains constant (equal to $\#|\pi_0(Y_x,y)|$). Now fix the base-point y. Let U be the image of $\pi_1(Y,y)$ in $\pi_1(X,x)$, and let $p\colon (X',x') \to (X,x)$ be the covering space associated to it. This covering has degree $\#|\pi_0(Y_x,y)|$, and, by Lemma A.8, there is a unique factorization $f = p \circ g$. Clearly, g is geometrically connected. It is also easy to check that, any other factorization $f = p' \circ g'$ in which p' is finite étale and g geometrically connected is uniquely isomorphic to the factorization $f = p \circ g$ (to see this, consider the open subgroup of $\pi_1(X,x)$ corresponding to p'. It should be contained in U, and should have the same index; so it must be equal to U). We have proved so far the existence and uniqueness of the factorization. We now prove that g satisfies **A**. By the very

construction of g, the axiom **A** is satisfied at the specific base-point y. But, by uniqueness, changing y does not change the factorization (up to isomorphism); so the exactness of the above sequence is valid for any other choice of the base point as well.
That the factorization commutes with base change is trivial. □

A.3. **Fibrations and descent.** Fibrations behave well with respect to fppf descent:

Theorem A.10. *Let $f\colon Y \to X$ be a morphism of algebraic spaces. Assume $a\colon V \to X$ is a (surjective) morphism that is a universally effective descent morphism for finite étale morphisms. Further, assume a is either universally open or universally closed. If the base extension of f along a is a fibration, then so is f. (Recall that 'universally' means that every base extension of a is also an effective descent morphism.)*

Proof. We may assume X and Y are connected. Let $W = Y \times_X V$, and let $r\colon W \to V$ be the projection map.

$$\begin{array}{ccc} W & \xrightarrow{b} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[a]{} & X \end{array}$$

By Proposition A.9, there is a unique (up to a unique isomorphism) factorization $g = q \circ t$ such that q is finite étale, and t has geometrically connected fibers. The uniqueness of this factorization, plus the fact that it is invariant under base change, imply that it satisfies descent condition relative to the map a (namely, by repeating the same argument with $V \times_X V$ and $V \times_X V \times_X V$ instead of V). Therefore, since a is a universally effective descent morphism, the factorization descends to a factorization for f. Hence, we are reduced to the case where f has geometrically connected fibers. In this situation, the exact sequence (3) implies that $g\colon W \to V$ is a disjoint union of geometrically connected morphisms that satisfy **A**.

To prove the exactness of (4) for f, we can now use the geometric translation of the exactness as in Proposition A.7. Let $Y' \to Y$ be a (connected) covering space of Y. Pick a geometric point x for X, and assume $Y' \to Y$ has a section over Y_x. We want to show that Y' descends uniquely to a covering space X' of X. Pick a connected component V_0 of V whose image in X contains x, and let W_0 be the (unique) connected component of W above it. Pick a geometric point v in V_0 lying above x. Let $W' = W \times_Y Y'$ be the pull back of Y' to W, and let W_0' be its restriction to W_0. Then, the restrictions of W_0' to $W_{0,v}$ has a section. Hence, W_0' descends (uniquely) to a covering space V_0' of V_0. This, in particular, implies that, if we replace x by any other point in the image of V_0, then $Y' \to Y$ has a section over Y_x. Therefore, if V_1 is another connected component of V whose image in X intersects that of V_0, then W_1' will descend (uniquely) to a covering space V_1' of V_1. We can continue this argument inductively and, using the fact that X is connected and the map $V \to X$ is open (or closed), we see that any point in X can be eventually reached after finitely many steps. Therefore, we have shown that W' descends uniquely to a covering space V' of V.

Repeating the same argument with $V \times_X V$ and $V \times_X V \times_X V$ instead of V, and using the uniqueness of descent, we see that V' satisfy the descent condition relative to a, so it descends to to a covering space X' of X. We claim that X' does the job. Let Y'' be the pull back of X' to Y. We claim that Y'' is isomorphic to Y'. To do so, it is enough to show that their pull backs to W are isomorphic (because, $b\colon W \to Y$ is an effective descent morphism). But this is true by the construction of X'. It only remains to prove that descent is unique; namely, that if X' and X'' are two covers of X whose pull backs Y' and Y'' to Y are isomorphic, then they are isomorphic. It is enough to show that pull backs V' and V'' of X' and X'' to V are isomorphic. But, since $g\colon W \to V$ is a disjoint union of geometrically connected morphisms that satisfy **A**, it is enough to prove that the pull backs W' and W'' of V' and V'' to W are isomorphic (Prposition A.7). But, these two are isomorphic to pull backs of Y' and Y'', which are already isomorphic over Y, to W. The proof is now complete. □

The following classes of morphisms satisfy the requirements of the Theorem ([4], Exposé IX):

- fppf.
- Proper, surjective and of finite presentation.
- Surjective, universally open and of finite type (target locally Noetherian).

A.4. **Fibrations for algebraic stacks.** The definition of a fibration and the arguments of the previous sections are quite formal and can be generalized to algebraic stacks.

Definition A.11. We say that a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of algebraic stacks is a *fibration* if for any choice of base points, the sequence (2) is exact.

In this definition choice of base points y and x should be made so that $f(y)$ is 2-isomorphic to x. A clarification here is in order: A pointed morphism between pointed stacks $(\mathcal{Y}, y)$ and $(\mathcal{X}, x)$ consists of a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ together with a transformation $\phi\colon x \rightsquigarrow f(y)$. We do need to have the extra data ϕ in order to be able to form the sequence (2). However, the exactness of this sequence is independent of the choice of ϕ.

When f is a representable morphism, then f being a fibration is equivalent to its base extension via every $X \to \mathcal{X}$ being a fibration, where X is a scheme (or an algebraic space). In fact, by Theorem A.10, to check that f is a fibration we can pick a flat chart $X \to \mathcal{X}$ and check whether the base extension of f to X is a fibration. This implies the following

Theorem A.12. *Let $R \overset{s,t}{\rightrightarrows} X$ be a flat and locally of finite presentation groupoid so that $R \to X \times X$ is separated and quasi compact (these conditions ensure that the quotient stack is algebraic). Let $\mathcal{X} = [X/R]$ be the corresponding quotient stack. Then, for $X \to \mathcal{X}$ to be a fibration it is necessary and sufficient that s (and t) be fibrations. In particular, let $G \to S$ be a group space that is flat, separated, locally of finite presentation and a fibration over S. Assume G acts on an algebraic space X over S. Then, the quotient map $X \to [X/G]$ is a fibration.*

References

1. K. Behrend, *Derived l-adic categories for algebraic stacks*, To appear in Memoirs of the AMS.
2. K. Behrend and B. Noohi, *Uniformization of Deligne-Mumford curves*, Preprint.
3. A. Grothendieck et al, *Schémas en Groupes*, Lecture Notes in Mathematics, no. 151,152,153, Springer-Verlag, Berlin-New York, 1962-64.

4. A. Grothendieck, *Revêtments étales et Groupe Fondamental*, Lecture Notes in Mathematics, no. 224, Springer-Verlag, Berlin-New York, 1971.
5. A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique IV*, IHES, Publ. Math. (1966), no. 28.
6. S. Keel and S. Mori, *Quorients by groupoids*, Ann. of Math. **145** (1997), no. 1, 193–213.
7. D. Knutson, *Algebraic Spaces*, Lecture Notes in Mathematics, no. 203, Springer-Verlag, 1971.
8. J. Kollar, *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) **145** (1997), no. 1, 33–79.
9. G. Laumon and L. Moret-Bailly, *Champs Algébriques*, Springer-Verlag, 1999.
10. D. Mumford, *Geometric Invariant Theory*, Springer-Verlag.
11. B. Noohi, *Fundamental groups of algebraic stacks*, Ph.D. thesis, M.I.T., 2000.
12. V. Zoonekynd, *The fundamental group of an algebraic stack*, math.AG/0111071.