FOUNDATIONS OF TOPOLOGICAL STACKS I

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Abstract. This is the first in a series of papers devoted to foundations of topological stacks.

We begin developing a homotopy theory for topological stacks along the lines of classical homotopy theory of topological spaces. In this paper we go as far as introducing the homotopy groups and establishing their basic properties. We also develop a Galois theory of covering spaces for a (locally connected semilocally 1-connected) topological stack. Built into the Galois theory is a method for determining the stacky structure (i.e., inertia groups) of covering stacks. As a consequence, we get for free a characterization of topological stacks that are quotients of topological spaces by discrete group actions. For example, this give a handy characterization of good orbifolds.

Orbifolds, *graphs of groups*, and *complexes of groups* are examples of topological (Deligne-Mumford) stacks. We also show that any algebraic stack (of finite type over $\mathbb{C}$) gives rise to a topological stack. We also prove a Riemann Existence Theorem for stacks. In particular, the algebraic fundamental group of an algebraic stack over $\mathbb{C}$ is isomorphic to the profinite completion of the fundamental group of its underlying topological stack.

The next paper in the series concerns function stacks (in particular loop stacks) and fibrations of topological stacks. This is the first in a series of papers devoted to foundations of topological stacks.

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1. Introduction

The theory of algebraic stacks was invented by Deligne and Mumford (largely based on ideas of Grothendieck) and extended later on by Artin, and has ever since become an indispensable part of algebraic geometry.

Topological stacks, however, have only appeared in heuristic arguments here and there, and no solid theory for them exists in the literature.[1] Since topological stacks carry the underlying "topology" of their ubiquitous algebraic counterparts, need for a systematic development for the homotopy theory (and, for that matter, measure theory, differential geometry and so on) seems inevitable. Furthermore, it is expected that (a substantial portion of) the theories of topological and differential groupoids (e.g. works of Weinstein, Moerdijk and others) to find their natural home in the theory of topological stacks.

I should point out that certain special classes of topological stack have already been around for decades, under secret names of *graphs of groups* (Serre, Bass), and *orbifolds* (Thurston). The theory developed in this paper brings together these two theories under the general umbrella of topological stacks.

My goal has been to set up a machinery that is general enough so it can be applied in variety of situations. The slogan is that, having chosen the correct definitions, topological stacks can be treated pretty much the same way as topological spaces and, modulo keeping track of certain extra structures that are bundled with them (I will try to explain this below), we can do with topological stack what we do with usual topological spaces (e.g. homotopy and (co)homology theories, differential geometry, measure theory and so on).

► Where do topological stacks come from?

Topological stacks provides a unified way to treat *equivariant* problems, as if they were non-equivariant: in most situations, one can formulate an equivariant problem about a G-space X as a problem about the quotient stack $[X/G]$. This point of view also provide a better functorial grasp on the problems, especially in situations where there is a change of groups involved.

The theory of topological stacks is a natural framework to study topological groupoids, and the homotopy theory developed here enables us to apply the machinery of algebraic topology in the study of topological groupoids. Morita invariant phenomena belong naturally to the realm of topological stacks.

Also, as in algebraic geometry, topological stacks can be useful in the study of moduli problems (e.g. classifying spaces etc.). Quotient constructions, especially in the presence of fixed points, are more naturally performed in the framework of stacks. The pathological behavior of quotient spaces, as well as the enormous loss of information, is well taken care of if we use the stacky approach.

Topological stacks can also appear as "underlying spaces" of algebraic (or differential, analytic etc.) stacks, much the same way that topological space appear as underlying spaces of algebraic varieties (or manifolds, complex manifolds etc.). The homotopy theoretical properties of the underlying topological stack is bound to play an important role in the study of the original algebraic stack.

► How to think about topological stacks?

[1]There is a paper by D. Metzler [Me] which has a bit of overlap (in some definitions) with this paper, but has a different focus.

Philosophically speaking, if one day it was declared that *set* theory, as a foundation for mathematics, should be thrown away and be replaced by *groupoid* theory, then we would have to replace topological spaces, complex manifolds, schemes and so on, by topological stacks, analytic stacks, algebraic stacks and so on!

Intuitively speaking, one way to think of a topological stack $\mathcal{X}$ is to think of a "topological space" in which a point is no longer a single point, but a cluster of points that are equivalent to each other.[2] There are "equivalences" between pairs of points in such a cluster, *possibly more than one for each pair*, and we would like to keep track of these equivalences; that is why we build them into the structure of our topological stack. If we chose to actually identify all the points in each cluster, we end up with an honest topological space X_{mod}, called the *coarse moduli space* of $\mathcal{X}$. If we make a choice of a point x in a cluster, it comes with an additional structure: the group of equivalences from x to itself. This is called the *inertia group* of x, and is denoted by I_x. Different points in the same cluster have isomorphic inertia groups. A very rough intuitive picture for $\mathcal{X}$ would be then to think of it as the collection of I_x bundled together along the topological space X_{mod}.

Of course, this picture is extremely handicapped, and one should be very cautious as there could occur quite a lot of pathological phenomena that are overlooked in this simplistic description. One is advised to work out a whole lot of examples (especially pathological ones) so as to adjust the old fashioned topological intuition to this situation.

An essential part of the intuition required to understand a stack is based on the following meta-mathematical principle: never identify equivalent (in a loose sense of the term) objects, only remember that they *can* be identified, and remember *the ways* they can be identified (i.e. remember all the equivalences between them).

Let me give a very simple example. Take a (discrete) set X, and let G be a discrete group acting on it. We would like to define the quotient of this action. The old fashioned way to do this is to look at the set X/G of orbits. That is, a given pair of points x and y in X are identified, if there is a $g \in G$ sending x to y. This construction, however, violates the principle mentioned in the previous paragraph. So let us modify the construction a little bit. What we want, instead of identifying x and y, is to remember that they *can* be identified via g. To do so, we draw an arrow from x to y, and put a label g on it. So now we have a set X with a collection of arrows between them, labeled by elements of G. It is easy to see that there is a natural way to compose the arrows. So we have actually constructed a category whose set of objects is X. This category is indeed a groupoid, and is sometimes referred to as the *translation groupoid* of the action of G on X. We think of this groupoid as the quotient of the action of G on X, and denote it by $[X/G]$. This is a baby example of a *quotient stack*.

Of course this is a discrete example and may seem not so interesting, but recall that Grothendieck tells us that any "space" X (e.g. topological space, differential manifold, scheme and so on) is just a collection of *sets*, coexisting in a compatible way, i.e. in the form of a sheaf. For instance, the information carried by a topological space X is completely captured by the sheaf $\mathsf{Top} \to \mathsf{Set}$ it represents (Yoneda lemma). So, after all, everything boils down to sets. What if we want to have

[2]To make our simplistic description closer to reality, we should consider also W-valued points for W an arbitrary topological space. These are maps from W to $\mathcal{X}$.

groupoids instead of sets (we just saw how groupoids turned up in the construction of our discrete quotient stack)?

The groupoid version of a sheaf is what is called a *stack*. So, a stack to a (discrete) groupoid is what a sheaf is to a set. The axioms of a stack are jazzed up versions of the axioms of sheaves (we have one more axiom actually), and they are sometimes called *descent conditions*. They are to be thought of as local-to-global conditions. The following examples explains what descent conditions mean. Assume $\mathcal{X}$ is a stack over the category Top of topological space, and let W be a topological space. Suppose we are given an open covering $\{W_i\}_{i\in I}$ of W, and for each i we are given a map $f_i\colon W_i \to \mathcal{X}$. Assume over each double intersection W_{ij}, f_i can be identified with f_j (but, by the principle mentioned above, we will not say that f_i is equal to f_j on W_{ij}), and we record this identification by giving it a name $\varphi_{ji}\colon f_i \Rightarrow f_j$. To make things as compatible as possible, we require that φ_{ij} is the inverse of φ_{ji}, and that over a triple intersection W_{ijk} we have $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ (cocycle condition). Having made all these provisions, it is natural to expect that the maps f_i can be glued along the identification φ_{ij} to form a global map $f\colon W \to \mathcal{X}$, and we want that to happen in an essentially unique way. This is what is called the *descent condition* or *stack condition*.

Of course, not every sheaf on Top is "topological enough" (i.e. comes from a topological space), so one should not expect that any old stack to be "topological enough" either. That is, simply imposing stack conditions is not enough for the purpose of doing topology. *Topological* stacks are stacks over Top which satisfy some extra conditions. This conditions appear in the form of existence of what we call a *chart* for the stack. We may perhaps also require that certain axioms be satisfied by this chart (these will be discussed in more detail in the text). Having imposed these conditions, one can *pretend* that the stack came from a "topological object", and the simplistic picture I gave a few paragraphs back is a way of visualizing that topological object.

► Structure of the paper

The paper is divided into two parts. The first part is more on the formal side and is devoted to setting up the basic definitions and constructions related to stacks over Top. No homotopy theory appears in Part I. In the Part II, we begin developing the homotopy theory. To keep the paper within bounds, we will not go that far into homotopy theory, and leave this task to the forthcoming papers in this series.

The sections are organized as follows:

I have collected some notations and conventions in Section 2. The reader who starts browsing from the middle of the paper and runs into some unfamiliar notation or terminology, or is unsure of the hypotheses being made all through the paper, may find this section helpful.

Section 3 contains some general facts about stacks over Grothendieck sites. The goal here is to set the notation, and collect some general results for the sake of future reference in the paper. It is by no means intended to be an introduction to stacks over Grothendieck sites – the reader is assumed to be familiar with the general machinery of stacks (see [Me], [LaMo]), and also with the language of 2-categories: stacks naturally form a 2-category and we will heavily exploit this feature.

There are five subsections to Section 3. Subsection 3.1 recalls the definition of 2-fiber products and 2-cartesian diagrams. Subsection 3.2 reviews the correspondence between groupoids (in the category of presheaves over a Grothendieck site C) and stacks over C. More precisely, given such a groupoid, one can construct its *quotient stack*. Conversely, any stack is of this form (up to equivalence). Subsection 3.3 is devoted to the definition of *image* of a morphism of stacks. The main feature of a stack $\mathcal{X}$ (as opposed to a sheaf) is that any point x on $\mathcal{X}$ comes with a group attached to it. This group is called the *inertia* group or the *isotropy group* of x, and is denoted by I_x. The inertia group is defined as the group of 2-isomorphisms from the map $x\colon * \to \mathcal{X}$ to itself (viewed in the 2-category of stacks). In examples where $\mathcal{X}$ is a quotient stack, I_x is isomorphic to the stabilizer group of x. The group I_x records how the point x has been "over-identified" with itself. So, for instance, if x has inertia group of order 5, we want to think of x as $\frac{1}{5}$th of a point!

Related to the notion of inertia group of a point are the notions of the inertia sheaf and the *residue gerbe* of a point. The inertia groups assemble together to form a global object $\mathcal{I}_{\mathcal{X}}$, called the *inertia stack*. The inertia stack comes with a map $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$, which makes it into a relative group object over $\mathcal{X}$. All these are discussed briefly in Subsection 3.4 of Section 3. In Subsection 3.5 we give a description of maps coming out of a quotient stack $[X/R]$ in terms of the groupoid $[R \rightrightarrows X]$ itself. The description is given in terms of maps coming out of X whose restriction to R satisfy certain cocycle conditions.

From Section 4 we narrow down to our favorite Grothendieck site, namely the category Top of compactly generated topological spaces. The Grothendieck topology is defined using the usual notion of covering by open subsets.[3] In Top we can be more concrete and talk about some specific topological features of stacks. Nevertheless, a stack over Top, as it is, is still to crude to do topology on. Some general considerations, however, can already be made at this level. There are three subsections to Section 4. In Subsection 4.1, we define the notion of a *representable* map between stacks, and discuss certain properties of representable maps such as local homeomorphism, open, closed and so on. In Subsection 4.2, we go over certain basic operations (intersection, union, closure, image, and inverse image) on substacks. A new feature of stacks over the site Top is that, to any such stack $\mathcal{X}$ we can associate an honest topological space X_{mod}, the *coarse moduli space* of $\mathcal{X}$. This is discussed in Subsection 4.3. There is natural map $\mathcal{X} \to X_{mod}$ which is universal among maps from $\mathcal{X}$ to topological spaces. So, in a way, X_{mod} is the best approximation of $\mathcal{X}$ by a topological space. We sometime call X_{mod} the *underlying topological space* of $\mathcal{X}$.

Sections 5 and 6 should really be thought of as subsections of Section 4. In Section 5 we quickly recall what a gerbe is, and look at certain specific features of gerbes over Top. Section 6 is devoted to an easy but very useful technical fact about representable maps. It is a criterion for verifying whether a map of stacks is representable. It says that, in order to check whether a map is representable, it is enough to do so after base extending the map along an epimorphism. This saves a lot of hassle later on where we have to frequently deal with representable maps.

In Section 7 we add flesh to the bone and introduce pretopological stack. These are stack over Top which admit a *chart*, that is, a representable epimorphism $p\colon X \to \mathcal{X}$ from a topological space X. Equivalently, a pretopological stack is

[3] I am convinced there is no advantage in taking other fancy topologies, unless someone proves me wrong.

equivalent to the quotient stack of a topological groupoid. The existence of a chart makes a stack more flexible, as we can use charts to transport things back to the world of topological spaces. However, pretopological stack are still too crude and not good enough to do topology on. The point is that, to introduce enough flexibility, so as to be able to carry out topological constructions, one has to impose some conditions on the chart $p\colon X \to \mathcal{X}$. This will be done in Section 13. But before that, we establish some general facts about pretopological stacks.

Different topological groupoids may give rise to the same quotient stack (up to equivalence). This phenomenon is usually referred to as *Morita equivalence* of groupoids. This is briefly discussed in Section 8.

A pretopological stack roughly corresponds to a Morita equivalence class of topological groupoids. Furthermore, by choosing suitable representatives in the corresponding Morita equivalence classes, a morphism of stacks can also be realized on the level of topological groupoids. An important question that arises is how to compute 2-fiber products of pretopological stack using their groupoid presentations. We answer this question in Section 9 by giving an explicit description of a topological groupoid that represents the 2-fiber product. This description is very useful when we want to prove that a 2-fiber product satisfies a certain property.

In Section 10 we go back to inertia sheaves and residue gerbes of points on a stack, now with the additional assumption that the stack admits a chart. In Section 11 we look at the coarse moduli space of pretopological stacks. It turns out that the coarse moduli space X_{mod} of a pretopological stack $\mathcal{X}$ is reasonably well-behaved if we assume that $\mathcal{X}$ admits a chart $p\colon X \to \mathcal{X}$ with p an *open* map. In this case, we prove the invariance of coarse moduli space under base change (Corollary 11.4).

There is an alternative description of the quotient stack of a topological groupoid in terms of torsors. This is discussed in Section 12. This explicit description of a quotient stack comes handy sometimes. It also explain why quotient stacks are related to classifying spaces. For example, when G is a topological group acting trivially on a point, the quotient stack, denoted by BG exactly classifies G-torsors.

This is the end of Part I.

In Section 13 we introduce topological stacks. These are our main objects of interest, and the ones on which we can perform enough topological constructions to have a reasonable homotopy theory. The topological stack are essentially pretopological stacks which admit a chart $p\colon X \to \mathcal{X}$ that is "nicely behaved". To make sense of what we mean by "nicely behaved", in Subsection 13.1 we introduce the notion of *local fibration*. This is defined to be a class of **LF** of continuous maps satisfying certain axioms. The axioms are modeled on the usual notion of a local fibration; that is, maps $f\colon X \to Y$ such that, after passing to appropriate open covers of X and Y, become a union of fibrations. Local Serre fibrations, local Hurewic fibrations and locally cartesian maps are examples of such a class **LF**.

For any choice of **LF** we have the corresponding theory of topological stacks, and the properties of **LF** reflect the properties of the corresponding category of topological stacks. We can use this to adjust our class of local fibrations to ensure that the corresponding topological stacks have the desired properties.

The role played by the notion of fibration becomes more apparent in Section 16 where we prove some gluing lemmas for topological stack. Roughly speaking, the point is that, in the category of pretopological stack gluing along *closed* substacks is problematic. For instance, very simple push-out diagrams in the category of topological spaces may no longer be push-out diagrams when viewed in the category of pretopological stacks. In particular, it is not possible do define homotopy groups of pretopological stacks (because, for example, a join of two circles does not satisfy the expected universal property when viewed in the category of pretopological stacks). However, once we impose the fibrancy condition on charts, the push-out diagrams start behaving better, provided we put certain *local cofibrancy* conditions on the arrows in the diagram. The bottom line is, more restrictions on the charts will result in higher flexibility in performing push-outs.

Parallel to the theory of Deligne-Mumford stacks in algebraic geometry, we develop a theory of Deligne-Mumford topological stacks. This is done in Section 14. A *weak Deligne-Mumford* topological stack is defined to be a topological stack that admits a chart $p\colon X \to \mathcal{X}$ that is a local homeomorphism. These are a bit too general and not as well-behaved as they are expected to be (as will be seen in some examples in the text). The good notion is that of a *Deligne-Mumford* topological stack; it is, by definition, a weak Deligne-Mumford stack that is locally a quotient stack of a properly discontinuous action (Definition 14.2). We will not make any finiteness assumptions on the stabilizers.

Dropping the finiteness condition in the definition of a Deligne-Mumford stack, however, causes some problems with existence of 2-fiber product, i.e. they may no longer be Deligne-Mumford. In Section 15 we see that, under very mild conditions, 2-fiber products of Deligne-Mumford topological stacks will again be Deligne-Mumford stacks (Corollary 15.8).

In Section 17 we start setting up the basic homotopy theory of topological stacks. We define homotopies between maps, and homotopy group of topological stacks. All the structures carried by homotopy groups of topological spaces (e.g. action of π_1 on higher homotopy groups, Whitehead products and so on) can be carried over to topological stacks. The homotopy groups of topological stacks, however, carry certain extra structure. More precisely, for a point x on a topological stack $\mathcal{X}$, we have natural group homomorphisms

$$\pi_n(\Gamma_x, x) \to \pi_n(\mathcal{X}, x),$$

where Γ_x is the residue gerbe at x. The importance of these maps is that they relate local invariants (i.e. residue gerbes) to global invariants (homotopy groups). When $n = 1$, $\pi_1(\Gamma_x)$ is isomorphic to the inertia group I_x at x. Therefore, we have natural maps $I_x \to \pi_1(\mathcal{X}, x)$. Note that the left hand side depends on the base point, while the right hand side does not (up to isomorphism). We discuss this map in some detail in Section 18.

In Subsection 18.2 of Section 18, we develop a Galois theory of covering spaces for topological stack and show that every locally path connected semilocally 1-connected topological stack has a universal cover (Corollary 18.20). In fact, we prove an equivalence between the category of $\pi_1(\mathcal{X})$-sets and the category of covering spaces of $\mathcal{X}$ (similar to Grothendieck's theory in SGA1). The covering theory developed here generalizes the covering theory of orbifolds developed by Thurston in [Th] and the covering theory of graphs of groups developed by Bass in [Ba].

In Subsection 18.3 of Section 18, we investigate the role played by the maps $I_x \to \pi_1(\mathcal{X}, x)$ mentioned above in the Galois theory of covering spaces. One main result is that, a Deligne-Mumford stack is globally a quotient of a (properly discontinuous) group action if and only if all the maps $I_x \to \pi_1(\mathcal{X}, x)$ are injective (Theorem 18.24).

This is a very useful and practical criterion, as the maps $I_x \to \pi_1(\mathcal{X}, x)$ are usually very easy to compute. This, in particular, gives a necessary and sufficient condition for an orbifold to be a *good orbifold* (in the sense of Thurston). In a separate paper [No2] we show how the maps $I_x \to \pi_1(\mathcal{X}, x)$ can be used to compute the fundamental group of the coarse moduli space X_{mod}. Essentially, the idea is that $\pi_1(\mathcal{X})$ is obtained by killing the images of all the maps $I_x \to \pi_1(\mathcal{X}, x)$, for various points x.[4]

In Section 19 we look at some examples of topological stacks. There are five subsections. In Subsection 19.1 we collect some pathological examples. In Subsection 19.2 we talk a little bit about *topological* gerbes. In Subsection 19.3 we show that Thurston's orbifolds are Deligne-Mumford topological stacks. In Subsection 19.4 we look at weighted projective lines. In Subsection 19.5 we show that graphs of groups are also Deligne-Mumford stacks. We point out how certain general results of Serre and Bass are easy consequences of the homotopy theory of topological stacks.

Finally, in Section 20, we construct a functor from the 2-category of algebraic stacks (of finite type over $\mathbb{C}$) to the 2-category of topological stacks. We show that, under this functor, Deligne-Mumford stacks go to Deligne-Mumford topological stacks. We also prove a Riemann existence for algebraic stack. In particular we deduce that, the algebraic fundamental group of an algebraic stack (of finite type over $\mathbb{C}$) is isomorphic to the profinite completion of the (topological) fundamental group of the corresponding topological stack.

Acknowledgment I would like to thank Kai Behrend and Angelo Vistoli, for sharing with me their ideas, as well as their personal notes on topological stack. I also thank Angelo Vistoli for inviting me to visit Bologna, during which we had interesting discussions about topological stacks. I would like to thank Rick Jardine for always being helpful and supportive. This paper was written during my Imperial Oil post-doctoral fellowship at the University of Western Ontario.

[4]This, in particular, gives a formula for computing the fundamental group of the the (coarse) quotient of a topological group acting on a topological space (possibly with fixed points).

2. Notation and conventions

For the convenience of the reader, in this section we collect some of the notations and conventions used throughout the paper, and fix some terminology. These are mostly items that are not explicitly mentioned during the text.

Throughout the paper, all topological spaces are assumed to be compactly generated.

We use calligraphic symbols ($\mathcal{X}$,$\mathcal{Y}$,$\mathcal{Z}$,...) for stacks, categories fibered in groupoids and so on, and Roman symbols (X,Y,Z,...) for spaces, sheaves and so on.

We fix a final object $*$, *the* point, in the category Top of topological spaces. Except for in Section 3.2, whenever we talk about a *point* x in a stack $\mathcal{X}$ what we mean is a morphism $x\colon * \to \mathcal{X}$ of stacks.

For an object W in a Grothendieck site C, we use the same notation W for the sheaf associated to the presheaf represented by W, and also for the corresponding stack. For the presheaf represented by W we use W_{pre}.

In the 2-category of stacks, the 2-isomorphism are synonymously referred to as *transformations*, *equivalences* or *identifications*. Given a pair $f,g\colon \mathcal{Y} \to \mathcal{X}$ of morphisms of stacks, we use the notation $f \overset{\varphi}{\Rightarrow} g$ or $\varphi\colon f \Rightarrow g$ to denote a 2-isomorphism between them. For ($*$-valued) points x and y in a stack, we use the alternative notation $x \rightsquigarrow y$ for an identification to emphasis the resemblance with a path.

The 1-morphisms in a 2-category are referred to as *morphisms* or *maps*.

Given a pair of morphisms of stacks $f\colon \mathcal{A} \to \mathcal{X}$ and $p\colon \mathcal{Y} \to \mathcal{X}$, by a *lift* of f to $\mathcal{Y}$ we mean a morphism $\tilde{f}\colon \mathcal{A} \to \mathcal{Y}$ together with an identification $\varphi\colon f \Rightarrow p \circ \tilde{f}$ as in the following 2-cell:

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "p"] \\ \mathcal{A} \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\Uparrow\varphi"] & \mathcal{X} \end{tikzcd}$$

When α and β are elements in some groupoid (e.g. groupoid of sections of a stack) such that $t(\alpha) = s(\beta)$, we use the multiplicative notation $\alpha \cdot \beta$ for their composition. We do the same with composition of 2-isomorphism.

A *substack* is always assumed to be saturated and full (Definition 3.8).

In some sloppy moments, we call an *isomorphism* (of groupoids/stacks) what should really be called an *equivalence*.

We use the notation $\mathcal{F} \mapsto \mathcal{F}^a$ for the stackification and sheafification functors.

The terms *étale* and *local homeomorphism* are synonymous. To be consistent with the classical terminology, we use the term "covering space/stack" in the topological setting for what we would call a"finite étale map" in the algebraic setting.

A groupoid object is denoted by $[R \rightrightarrows X]$, $[R \overset{s,t}{\rightrightarrows} X]$, or $[s,t\colon R \rightrightarrows X]$ and the symbols s and t are exclusively used for source and target maps. For a subset $U \subseteq X$, the *orbit* $t\big(s^{-1}(U)\big)$ of U is denoted by $\mathcal{O}(U)$.

When $[R \rightrightarrows X]$ is a groupoid object in a category (say Top) and $Y \to X$ is a map, we denote the corresponding *pull-back groupoid* by $[R|_Y \rightrightarrows Y]$, where $R|_Y := (Y \times Y) \times_{X \times X} R$.

Part 1. Preliminaries

3. Quick review of stacks over sites

In this section we collect a few facts about stacks that will be used throughout the paper. It should not be regarded as an introduction to stacks: the reader is assumed to be familiar with the notions of Grothendieck topology, sheaf and presheaf, category fibered in groupoids, stack and prestack, stackification and 2-category.

Throughout this section, we fix a category C with a Grothendieck topology on it. All categories fibered in groupoids will be over C. We will be sloppy and not distinguish between a presheaf and the corresponding category fibered in groupoids.

Let $\mathcal{X}$ be a category fibered in groupoids over C. For an object W in C, we have the corresponding **groupoid of sections** $\mathcal{X}(W)$. By Yoneda's lemma, there is a natural equivalence of groupoids

$$\mathrm{Hom}(W_{pre}, \mathcal{X}) \cong \mathcal{X}(W),$$

where the left hand side is computed in the 2-category of categories fibered in groupoids over C. Here, W_{pre} stands for the category fibered in groupoids associated to the presheaf represented by W.

The assignment $W \mapsto \mathcal{X}(W)$ gives rise to a *lax* presheaf of groupoids.[5] Conversely, given a (lax) presheaf of groupoids, one can construct the associated category fibered in groupoids. For the sake of brevity, we usually specify a category fibered in groupoids $\mathcal{X}$ by its groupoids of sections $\mathcal{X}(W)$.

3.1. **2-fiber products.**

There is a notion of 2-fiber product for categories fibered in groupoids. Consider a diagram

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"'] & \mathcal{X} \end{tikzcd}$$

where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are categories fibered in groupoids over C. The **2-fiber product** $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ is defined to be the category given by

$$\mathrm{Ob}(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}) = \left\{ \begin{array}{ccl} (y, z, \alpha) & | & y \in \mathrm{Ob}\, \mathcal{Y}, z \in \mathrm{Ob}\, \mathcal{Z}, \text{ s.t. } p_{\mathcal{Y}}(y) = p_{\mathcal{Z}}(z) = c \in \mathrm{Ob}\, \mathsf{C}; \\ & & \alpha \colon f(y) \to g(z) \text{ an arrow in } \mathcal{X}(c). \end{array} \right\}$$

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[5]This lax presheaf can be strictified, in the sense that it is "equivalent" to the strict presheaf $W \to \mathrm{Hom}(W_{pre}, \mathcal{X})$. But we don't really care.

$$\mathrm{Mor}_{\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z}}\big((y_1,z_1,\alpha),(y_2,z_2,\beta)\big) = \left\{ (u,v) \;\middle|\; \begin{array}{l} u\colon y_1\to y_2,\ v\colon z_1\to z_2 \text{ s.t.:}\\ \bullet\ p_{\mathcal{Y}}(u)=p_{\mathcal{Z}}(v)\in \mathrm{Mor}\,\mathsf{C},\\ \bullet\ \text{In Mor }\mathcal{X}\text{ we have,}\\ \qquad\qquad \begin{array}{ccc} f(y_1) & \xrightarrow{f(u)} & f(y_2)\\ \alpha\downarrow & \circlearrowleft & \downarrow\beta \\ g(z_1) & \xrightarrow[g(u)]{} & g(z_2)\end{array}\end{array}\right\}$$

Here $p_{\mathcal{Y}}\colon \mathcal{Y}\to\mathsf{C}$ stands for the structure map which makes $\mathcal{Y}$ fibered over C.

It is easily checked that $\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z}$ is again fibered in groupoids over C. The 2-fiber product should be viewed as a homotopy fiber product. When $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are stacks, the 2-fiber product $\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z}$ is again a 2-stack. We will not prove this here, but the idea is that the process of stackification consists of successive applications of homotopy limits combined with *filtered* homotopy colimits; in particular, it commutes with finite homotopy limits.

The 2-fiber product has the following universal property:

> For any category fibered in groupoids $\mathcal{W}$, the natural functor from the groupoid $\mathrm{Hom}(\mathcal{W},\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z})$ to the groupoid of triples (u,v,α), where $u\colon\mathcal{W}\to\mathcal{Y}$ and $v\colon\mathcal{W}\to\mathcal{Z}$ are morphisms and $\alpha\colon f\circ u\Rightarrow g\circ v$ a 2-isomorphism (e.g. a transformation of functors relative to C), is an equivalence.

We say that a diagram

$$\begin{tikzcd} \mathcal{W} \arrow[r, "u"] \arrow[d, "v"'] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"'] & \mathcal{X} \end{tikzcd}$$

of categories fibered in groupoids (or stacks) is **2-commutative**, if there is a 2-isomorphism $\alpha\colon f\circ u\Rightarrow g\circ v$. A 2-commutative diagram is sometimes called a **2-cell** and is denoted by

$$\begin{tikzcd} \mathcal{W} \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\swArrow\alpha"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X} \arrow[r, "g"'] & \mathcal{X} \end{tikzcd}$$

A 2-commutative diagram as above is called **2-cartesian**, if the induced map $(u,v,\alpha)\colon\mathcal{W}\to\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z}$ is an equivalence.

3.2. **Groupoids vs. stacks.** Let G be a (discrete) groupoid, and let X be a set. Given a map of sets $f\colon X\to\mathrm{Ob}(G)$, we can pull back the groupoid structure over to X by setting $\mathrm{Hom}(x,y):=\mathrm{Hom}_G\big(p(x),p(y)\big)$, for every $x,y\in X$. We obtain a groupoid H, with $\mathrm{Ob}(H)=X$. The groupoid H maps fully faithfully to G; this map is an equivalence if and only if f is surjective. The set $R=\mathrm{Mor}(H)$ of morphisms of H fits in the following 2-cartesian diagram of groupoids:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & G \arrow[d, "\Delta"] \\ X \times X \arrow[r, "(f,f)"'] & G \times G \end{tikzcd}$$

which can also be thought of as an alternative definition for R (hence for the pull back groupoid H).

We can do the same construction globally. Let $\mathcal{X}$ be a category fibered in groupoids over C (or a presheaf of groupoids over C, if you wish), and let X be a presheaf of sets (viewed as a category fibered in groupoids). Let $f\colon X \to \mathcal{X}$ be a map of categories fibered over C. Define the presheaf (of sets) R by the following 2-cartesian diagram diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ X \times X \arrow[r, "(f,f)"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Equivalently, $R \cong X \times_{\mathcal{X}} X$. This gives us a groupoid object $[R \rightrightarrows X]$ in the category of presheaves of sets. For each $W \in \mathrm{Ob}\,\mathsf{C}$, the groupoid $[R(W) \rightrightarrows X(W)]$ is the groupoid induced from $\mathcal{X}(W)$ on the set $X(W)$ via the map $X(W) \to \mathrm{Ob}\,\mathcal{X}(W)$, exactly as we saw in the previous paragraph.

Notation. Let $[R \rightrightarrows X]$ be a groupoid object in the category of presheaves. We denote the corresponding (strict) presheaf of groupoids, and also the corresponding category fibered in groupoids, by $\lfloor X/R \rfloor$.

Definition 3.1. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibered in groupoids.

- f is called an **epimorphism**, if for every $W \in \mathrm{Ob}\,\mathsf{C}$ and every $y \in \mathrm{Ob}\,\mathcal{Y}(W)$, there exists a covering family $\{U_i \to W\}$ such that for every i, y_{U_i} is (equivalent to an element) in the image of $\mathcal{X}(U_i)$.
- f is called a **monomorphism**, if for every $W \in \mathrm{Ob}\,\mathsf{C}$, the induced map $\mathcal{X}(W) \to \mathcal{Y}(W)$ is fully faithful.

The following proposition is immediate.

Proposition 3.2. *Let $\mathcal{X}$ be a category fibered in groupoids, X a presheaf of sets (viewed as a category fibered in groupoids), and $f\colon X \to \mathcal{X}$ a morphism. Set $R = X \times_{\mathcal{X}} X$, and let $[R \rightrightarrows X]$ be the corresponding groupoid. Then we have the following.*

- **i**. *The natural map $\lfloor X/R \rfloor \to \mathcal{X}$ is a monomorphism. In particular, if $\mathcal{X}$ is a prestack, then $\lfloor X/R \rfloor$ is also a prestack.*
- **ii**. *If $f\colon X \to \mathcal{X}$ is an epimorphism, then the induced map $\lfloor X/R \rfloor \to \mathcal{X}$ induces an equivalence of stacks $\lfloor X/R \rfloor^a \to \mathcal{X}^a$, where $-^a$ stands for stackification.*

Definition 3.3. Let $[R \rightrightarrows X]$ be a groupoid object in the category of presheaves of groupoids over C. We define the **quotient stack** $[X/R]$ to be the stack asscociated to $\lfloor X/R \rfloor$, that is, $\lfloor X/R \rfloor^a$. When G is a presheaf of groups relative to a presheaf (of sets) X, we define $[X/G]$ to be the quotient stack of the trivial groupoid $[G \times X \rightrightarrows X]$. This is sometimes denoted by $B_X G$ (or simply $\mathcal{B}G$, if the base X is understood), and is called the **classifying stack** of G.

A special case of the Proposition 3.2 is the following.

Proposition 3.4. *Let $\mathcal{X}$ be a stack, and let $X \to \mathcal{X}$ be an epimorphism from a presheaf of sets X (viewed as a category fibered in groupoids) to $\mathcal{X}$. Consider the groupoids object $[R \rightrightarrows X]$ (in the category of presheaves of sets over C) defined by $R = X \times_{\mathcal{X}} X$. Then, we have a natural equivalence $[X/R] \xrightarrow{\sim} \mathcal{X}$.*

The following well-known fact is along the same lines of the previous proposition.

Proposition 3.5. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of categories fibered in groupoids that is both a monomorphism and an epimorphism. Then the natural map $\mathcal{Y}^a \xrightarrow{\sim} \mathcal{X}^a$ is an equivalence.*

The following facts also come handy sometimes.

Proposition 3.6. *Let $[R \rightrightarrows X]$ be a groupoid object in the category of sheaves over C. Then $\lfloor X/R \rfloor$ is a prestack.*

Proposition 3.7. *Let $\mathcal{X}$ be a prestack over C. Then the natural map $\mathcal{X} \to \mathcal{X}^a$ is a monomorphism.*

3.3. **Image of a morphism of stacks.**

Definition 3.8. Let $\mathcal{X}$ be a stack over C. By a **substack** of $\mathcal{X}$ we mean a subcategory $\mathcal{Y}$ (fibered in groupoids over C) of $\mathcal{X}$ such that for every $W \in \operatorname{Ob}\mathsf{C}$, the subcategory $\mathcal{Y}(W) \subseteq \mathcal{X}(W)$ is full and saturated (equivalently, $\mathcal{Y}$ is a full saturated subcategory of $\mathcal{X}$ as abstract categories). A subcategory is called *saturated* if whenever it contains an object then it contains the entire isomorphism class of that object.

Definition 3.9. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks. The **image** $\operatorname{im}(f)$ of f is defined to be the smallest substack of $\mathcal{X}$ through which f factors.

An alternative definition for $\operatorname{im}(f)$ is that it satisfies the following properties:

- **I1**. $\operatorname{im}(f)$ is a subcategory fibered in groupoids of $\mathcal{X}$ (not a priori full or saturated) through which f factors;
- **I2**. $f\colon Y \to \operatorname{im}(f)$ is an epimorphism;
- **I3**. $\operatorname{im}(f)$ is the largest subcategory fibered in groupoids of $\mathcal{X}$ satisfying (**I1**) and (**I2**).

Proposition 3.10.

- **i**. *Let $g\colon \mathcal{Z} \to \mathcal{Y}$ and $f\colon \mathcal{Y} \to \mathcal{X}$ be morphisms of stacks. Then* $\operatorname{im}(f|_{\operatorname{im}(g)}) = \operatorname{im}(f \circ g)$.
- **ii**. $f\colon \mathcal{Y} \to \mathcal{X}$ *is an epimorphism if and only if* $\operatorname{im}(f) = \mathcal{X}$.

We also have the following way of computing the image:

Proposition 3.11. *Let $\mathcal{X}$ be a stack, and let $f\colon X \to \mathcal{X}$ be a map from a presheaf of sets X to $\mathcal{X}$. Let $[R \rightrightarrows X]$ be the corresponding groupoid in presheaves of sets (see Section 3.2). Then, we have a natural equivalence $[X/R] \to \operatorname{im}(f)$ (see Definition 3.3).*

Proof. By Proposition 3.2, we have a monomorphism $\lfloor X/R \rfloor \to \mathcal{X}$. Let $\mathcal{I}$ be the image of this map. By Proposition 3.5, this induces an equivalence $[X/R] \to \mathcal{I}$. Since $X \to [X/R]$ is an epimorphism, we have $\mathrm{im}(f) = \mathcal{I}$ by Proposition 3.10**i**. $\square$

3.4. **Inertia groups and inertia stacks.** Let W be an object in the base category C. By a W**-valued point** of $\mathcal{X}$ we mean an object x in the groupoid $\mathcal{X}(W)$ (or the corresponding morphism $x\colon W \to \mathcal{X}$). For such a point x, we can talk about the group of automorphisms of x, viewed as an object in the groupoid $\mathcal{X}(W)$. If we think of x as a map $W \to \mathcal{X}$, then this is just the group of self-transformations of x. This group is called the **inertia group** of x, and is denoted by I_x. Other synonyms for inertia group are *stabilizer group*, *isotropy group* and *ramification group*. We may use these terms interchangeably here and there.

The inertia groups I_x of a W-valued point x is in fact the set of W-points of a sheaf of groups $\mathbb{I}_x$ relative to W_{pre}, which we call the **inertia presheaf** at x. As a presheaf over C, the inertia presheaf $\mathbb{I}_x$ is defined as follow:

$$\mathbb{I}_x(W') = \left\{ \begin{array}{ccl} (f,\alpha) & \mid & f\colon W' \to W \text{ a morphism;} \\ & & \alpha \;\; \text{an automorphism of } f^*(x) \in \mathrm{Ob}\,\mathcal{X}(W') \end{array} \right\}$$

The forgetful map makes $\mathbb{I}_x$ a sheaf of groups relative to W_{pre}. When W_{pre} is a sheaf, then so is $\mathbb{I}_x$. This is always the case when the topology on C is subcanonical.

Definition 3.12. Let $\mathcal{X}$ be a category fibered in groupoids over C. We define $\pi_0\mathcal{X}$ to be the sheaf associated to the presheaf of sets on C defined by

$$W \mapsto \{\text{isomorphism classes in } \mathcal{X}(W)\}.$$

There is a natural map $\mathcal{X} \to \pi_0\mathcal{X}$. This map is universal among maps $\mathcal{X} \to Y$, where Y is a sheaf over C. For any W-point x of $\mathcal{X}$, we can compose it with $\mathcal{X} \to \pi_0\mathcal{X}$ to obtain a W-point for $\pi_0\mathcal{X}$, which we denote by $\bar{x}$.

The inertia presheaves $\mathbb{I}_x$ measure the difference between $\mathcal{X}$ and $\pi_0\mathcal{X}$ in the following sense.

Proposition 3.13. *Let x be a W-point of $\mathcal{X}$. Then we have a natural 2-cartesian diagram*

$$\begin{tikzcd} \mathcal{B}_W\mathbb{I}_x \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ W \arrow[r, "\bar{x}"'] & \pi_0\mathcal{X} \end{tikzcd}$$

where $B_W\mathbb{I}_x = [W/\mathbb{I}_x]$ is the classifying stack of the inertia presheaf $\mathbb{I}_x$ (see Definition 3.3).

Proof. Let X_0 denote the presheaf

$$W \mapsto \{\text{isomorphism classes in } \mathcal{X}(W)\}.$$

It follows formally from the definitions that the following diagram is 2-cartesian

$$\begin{tikzcd} \lfloor W/\mathbb{I}_x \rfloor \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ W_{pre} \arrow[r] & X_0 \end{tikzcd}$$

The desired 2-cartesian diagram is the stackification of this diagram. $\square$

Definition 3.14. Let $\mathcal{X}$ be a stack over a site C, and let $x\colon W \to \mathcal{X}$ be a W-point. We define the **residue stack** Γ_x of $\mathcal{X}$ at x to be the image of x, that is $\Gamma_x = \mathrm{im}(x)$.

Using the definition of the image (Section 3.3) the following proposition is immediate

Lemma 3.15. *The following diagram is 2-cartesian:*

$$\begin{tikzcd} \mathrm{im}(x) \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{im}(\bar{x}) \arrow[r, hook] & \pi_0\mathcal{X} \end{tikzcd}$$

Corollary 3.16. *Notation being as above, assume $\bar{x}\colon W \to \pi_0\mathcal{X}$ is a monomorphism. Then we have a 2-cartesian diagram*

$$\begin{tikzcd} \Gamma_x \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ W \arrow[r, "\bar{x}"'] & \pi_0\mathcal{X} \end{tikzcd}$$

Combining Lemma 3.15 with Proposition 3.13, we get the following

Corollary 3.17. *We have a natural 2-cartesian diagram*

$$\begin{tikzcd} B_W\mathbb{I}_x \arrow[r] \arrow[d] & \Gamma_x \arrow[d] \\ W \arrow[r, "\bar{x}"'] & \pi_0\mathcal{X} \end{tikzcd}$$

In particular, when $\bar{x}\colon W \to \pi_0\mathcal{X}$ is a monomorphism, we have a natural equivalence $B_W\mathbb{I}_x \xrightarrow{\sim} \Gamma_x$.

The inertia sheaves can be assembled in a more global object. For a stack $\mathcal{X}$, we define its **inertia stack** $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$, by the following 2-fiber product:

$$\begin{tikzcd} \mathcal{I}_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

For an object W in C, the groupoid of sections $\mathcal{I}_{\mathcal{X}}(W)$ is naturally equivalent to the following groupoid:

$$\mathrm{Ob} = \left\{ \begin{array}{ccl} (x,\alpha) & | & x \in \mathrm{Ob}\,\mathcal{X}(W); \\ & & \alpha \ \text{ an automorphism of } x \end{array} \right\}$$

$$\mathrm{Mor}\big((x,\alpha),(x',\alpha')\big)=\big\{\gamma\in\mathrm{Mor}_{\mathcal{X}(W)}(x,x')\mid\alpha\cdot\gamma=\gamma\cdot\alpha'\big\}$$

The map $\mathcal{I}_{\mathcal{X}}\to\mathcal{X}$ corresponds the forgetful map.

The inertia stack and the inertia presheaf of a W-point x are related by the following 2-cartesian diagram:

$$\begin{tikzcd}\mathbb{I}_x \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ W \arrow[r, "x"'] & \mathcal{X}\end{tikzcd}$$

In particular, the groupoid of possible liftings of W to $\mathcal{I}_{\mathcal{X}}$ is equivalent to a set whose element are in natural bijection with the group of self-transformations of x (which is the same as the inertia group I_x). This is in fact true if W is any stack (or category fibered in groupoids) over C. In particular, when $W=\mathcal{X}$, we find that the set of sections (up to identification) of the map $\mathcal{I}_{\mathcal{X}}\to\mathcal{X}$ is in natural bijection with the group of self-transformations of the identity functor $\mathrm{id}\colon\mathcal{X}\to\mathcal{X}$.

We state the following proposition for future use. It is easy to prove.

Proposition 3.18. *Let $f\colon\mathcal{Y}\to\mathcal{X}$ be a monomorphism. Then we have a natural 2-cartesian diagram*

$$\begin{tikzcd}\mathcal{I}_{\mathcal{Y}} \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X}\end{tikzcd}$$

3.5. **Maps coming out of a quotient stack.** Let $\mathcal{X}=[X/R]$ be the quotient stack of a groupoid object (in presheaves of sets over C) $[s,t\colon R\rightrightarrows X]$, and let $\mathcal{Y}$ an arbitrary stack. The groupoid $\mathrm{Hom}_{\mathbf{St}}(\mathcal{X},\mathcal{Y})$ of maps from $\mathcal{X}$ to $\mathcal{Y}$ has a description in terms of cocycles. More precisely, define the groupoid $C(\mathcal{X},\mathcal{Y})$ as follows:

$$\mathrm{Ob}\,C(\mathcal{X},\mathcal{Y})=\left\{\begin{array}{ccl}(f,\varphi) & \mid & f\colon X\to\mathcal{Y},\ f\circ s\overset{\varphi}{\Rightarrow}f\circ t\\ & & \text{such that}\quad \begin{tikzcd}[column sep=small] f\circ d_0 \arrow[rr, Rightarrow, "\varphi\circ\mu"] \arrow[dr, Rightarrow, "\varphi\circ pr_1"'] & & f\circ d_2 \\ & f\circ d_1 \arrow[ur, Rightarrow, "\varphi\circ pr_2"'] & \end{tikzcd}\end{array}\right\}$$

$$\mathrm{Mor}_{C(\mathcal{X},\mathcal{Y})}\big((f,\varphi),(f',\varphi')\big)=\left\{\begin{array}{ccl}\alpha & \mid & f\overset{\alpha}{\Rightarrow}f' \text{ an identification}\\ & & \text{such that}\quad \begin{tikzcd} f\circ s \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, "\alpha\circ s"'] \arrow[dr, phantom, "\circlearrowleft"] & f\circ t \arrow[d, Rightarrow, "\alpha\circ t"] \\ f'\circ s \arrow[r, Rightarrow, "\varphi'"'] & f'\circ t\end{tikzcd}\end{array}\right\}$$

Here, $d_0,d_1,d_2\colon R\times_X R\to X$ are the obvious maps, $pr_1,pr_2\colon R\times_X R\to R$ are the projection maps, and $\mu\colon R\times_X R\to R$ is the multiplication.

Proposition 3.19. *The natural functor*

$$\Sigma\colon \mathrm{Hom}_{\mathbf{St}}(\mathcal{X},\mathcal{Y}) \to C(\mathcal{X},\mathcal{Y})$$

is an equivalence of groupoids.

Sketch of proof. We show how to construct an inverse Θ for Σ.

First, we claim that there is a presheaf of sets Y over C with an epimorphism $q\colon Y \to \mathcal{Y}$ such that, for every presheaf of sets X and every map $f\colon X \to \mathcal{Y}$, there exists a lift $\tilde{f}\colon X \to Y$. The presheaf Y is defined as follows:

$$W \mapsto \mathrm{Ob}\,\mathrm{Hom}(W_{pre},\mathcal{Y}).$$

It is left to the reader to verify that Y has the claimed property.

Set $T = Y \times_{\mathcal{Y}} Y$. By Proposition 3.4, we have an equivalence $[Y/T] \xrightarrow{\sim} \mathcal{Y}$. Now, given an object (f,φ) in $C(\mathcal{X},\mathcal{Y})$, pick a lift $\tilde{f} : X \to Y$ of f. The transformation $f \circ s \overset{\varphi}{\Rightarrow} f \circ t$ exactly translates as a map $R \to T$ (of presheaves of sets) for which the following diagram commutes:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & T \arrow[d] \\ X \times X \arrow[r] & Y \times Y \end{tikzcd}$$

The cocycle condition satisfied by φ also translates as saying that the map $R \to T$ makes $[R \rightrightarrows X] \to [T \rightrightarrows Y]$ a morphism of groupoids.

The induced map on the quotient stacks gives us the desired map $\Theta(f,\varphi)\colon \mathcal{X} \to \mathcal{Y}$. The effect of Θ on morphisms of $C(\mathcal{X},\mathcal{Y})$ is defined in a similar way. $\square$

4. Stacks over the topological site

From now on, all stacks are over the site Top defined below, unless otherwise specified.

Let Top be the category of compactly generated topological with continuous maps. We endow Top with a Grothendieck topology defined by taking the open coverings to be the usual open coverings of topological spaces. This is a subcanonical topology. That means, the presheaf W_{pre} represented by any $W \in \mathsf{Top}$ is indeed a sheaf (so we denote it again by W).

We denote the 2-category of stacks over Top by $\mathsf{St}_{\mathsf{Top}}$; it contains Top as a full sub-category (Yoneda).

Throughout the paper we fix a final object $*$ in Top.

Definition 4.1. By a **point** x of a stack $\mathcal{X}$ over Top we mean a $*$-valued point; that is, a map of stacks $x\colon * \to \mathcal{X}$. We sometimes abuse notation and denote this by $x \in \mathcal{X}$.

4.1. **Representable maps.** In $\mathsf{St}_{\mathsf{Top}}$, there are certain maps which behave much like maps of topological spaces. In particular, many properties of continuous maps of topological spaces makes sense for them.

Definition 4.2. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks. We say f is **representable**, if for any map $X \to \mathcal{X}$ from a topological space X, the fiber product $Y := X \times_{\mathcal{X}} \mathcal{Y}$ is (equivalent to) a topological space.

Proposition 4.3. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a representable morphism of stacks. Then,*

i. *for every stack $\mathcal{Z}$, the map*

$$\mathrm{Hom}_{\mathsf{St}}(\mathcal{Z},\mathcal{Y}) \to \mathrm{Hom}_{\mathsf{St}}(\mathcal{Z},\mathcal{X})$$

is faithful. In particular, for every topological space W, $\mathcal{Y}(W) \to \mathcal{X}(W)$ is faithful.

ii. *for every topological space W, and every W point $y\colon W \to \mathcal{Y}$, the natural group homomorphism $I_y \to I_{f(y)}$ is injective.*

Proof. Easy. □

Definition 4.4. Let **P** be a property of morphisms of topological spaces.

- We say **P** is **invariant under base change**, if for every map $f\colon Y \to X$ which is **P**, the base extension $f'\colon Y' \to X'$ of f along any $q\colon X' \to X$ is again **P**. We say **P** is **invariant under restriction**, if the above condition is satisfied when $q\colon X' \to X$ is an embedding (i.e. inclusion of a subspace).
- We say **P** is **local on the target**, if for every map $f\colon Y \to X$, the base extension $f'\colon Y' \to X'$ of f along a *surjective local homeomorphism* $q\colon X' \to X$ being **P** implies that f is **P**. (*Remark.* If we replace 'surjective local homeomorphism' by 'open covering' we get the same notion. If **P** is invariant under restriction, we could also replace 'surjective local homeomorphism' by 'epimorphism', arriving at the same notion.)

Remark that a map of topological spaces is an epimorphism if and only if it admits local sections.

Definition 4.5. Let **P** be a property of morphisms of topological spaces that is invariant under restriction and local on the target. We say a representable map $f\colon \mathcal{Y} \to \mathcal{X}$ of stacks is **P**, if there exists an epimorphism $q\colon X \to \mathcal{X}$ such that the the base extension $Y \to X$ of f along q is **P**.

If **P** in the above definition is also invariant under base extension, then it follows that for *every* $q\colon X \to \mathcal{X}$ the base extension $Y \to X$ is **P**.

Example 4.6.

1. Any of the following is invariant under base change and local on the target:

> *open, epimorphism, surjective, embedding* (see below for definition), *closed embedding, open embedding, local homeomorphism, covering map, finite fibers, discrete fibers.*

2. Any of the following is invariant under restriction and local on the target:

> *closed, closed onto image.*

Definition 4.7. We say a map $f\colon Y \to X$ of topological spaces is an **embedding**, if f induces a homeomorphism from Y onto the subspace $f(Y)$ of X. When the image of f is open, we say f is an **open embedding**. When the image of f is closed, we say f is a **closed embedding**. A composite of open and closed embeddings is called a **locally-closed embedding**. All these notions are invariant under base change and local on the target, so we can define them for representable maps of stacks (Definition 4.5).

Lemma 4.8. *Every embedding is a monomorphism.*

Proof. Easy. □

Definition 4.9. Let $\mathcal{X}'$ be a substack of $\mathcal{X}$. We say that $\mathcal{X}'$ is an **embedded substack** if the inclusion map $\mathcal{X}' \hookrightarrow \mathcal{X}$ is an embedding. **Open substack** and **closed substack** are defined in a similar fashion.

Note that the inclusion of a substack $\mathcal{X}' \hookrightarrow \mathcal{X}$ is not necessarily representable, so not every substack is embedded.

4.2. **Operations on substacks.** We will briefly go over the notions of *intersection*, *union*, *inverse image* and *image* for substacks of a stacks over Top. What we are really interested in the case where substacks are *embedded*.

Let $\mathcal{X}$ be a stack over Top, and let $\{\mathcal{X}_i\}_{i\in I}$ be a family of substacks. The **intersection** $\bigcap_{i\in I}\mathcal{X}_i$ is defined to be the intersection of the categories $\mathcal{X}_i$ in the set theoretic sets (note that substacks are assumed to be saturated and full). It is easy to see that the intersection of substacks is again a substack.

Let $\mathcal{X}'$ be a substack of $\mathcal{X}$ and let $\mathcal{Y} \to \mathcal{X}$ be a morphism of stacks. We define the **inverse image** $\mathcal{Y}'$ of $\mathcal{X}'$ to be the image (Definition 3.9) of the natural monomorphism $\mathcal{Y}\times_{\mathcal{X}}\mathcal{X}' \to \mathcal{Y}$. The inverse image is again a substack.

Lemma 4.10.

- **i**. *Taking inverse image commutes with intersection.*
- **ii**. *Intersection of* embedded *substacks is again embedded. Intersection of closed substacks is again a closed substack.*
- **iii**. *Inverse image of an embedded (resp. closed, open) substack under an arbitrary morphism of stacks is again an embedded (resp., closed, open) substack.*

Proof. Straightforward. □

Part (**ii**) of the lemma implies that, for any substack $\mathcal{X}'$ of a stack $\mathcal{X}$, there is a canonical (unique) smallest embedded substack $\mathcal{X}''$ of $\mathcal{X}$ that contains $\mathcal{X}'$. In general, given a collection of substacks $\{\mathcal{X}_i\}_{i\in I}$ of $\mathcal{X}$, there is a canonical (unique) smallest embedded substacks of $\mathcal{X}$ containing all the $\mathcal{X}_i$.

Similarly, given a closed substack $\mathcal{X}'$ of a stack $\mathcal{X}$, we define its **closure** $\bar{\mathcal{X}}'$ to be the intersection of all closed substacks containing $\mathcal{X}'$. The closure $\bar{\mathcal{X}}'$ is a closed substack of $\mathcal{X}$.

The **union** $\bigcup_{i\in I}\mathcal{X}_i$ of a family $\{\mathcal{X}_i\}_{i\in I}$ of embedded substacks of $\mathcal{X}$ is defined to be the smallest embedded substack of $\mathcal{X}$ containing all the $\mathcal{X}_i$. An alternative description for the union of substacks is given by the following lemma.

Lemma 4.11. *An object $x \in \mathcal{X}(W)$ is in $\bigcup_{i\in I}\mathcal{X}_i(W)$ if and only if the corresponding map $W \to \mathcal{X}$ has the property that $\bigcup_{i\in I} W_i = W$, where W_i is the inverse image of $\mathcal{X}_i$ in W (which is a subspace of W).*

Lemma 4.12.

- **i**. *Taking inverse image commutes with union.*
- **ii**. *Union of open substacks is again an open substack.*

Proof. Follows from Lemma 4.11. □

Finally, given a map $f\colon \mathcal{Y} \to \mathcal{X}$ of stacks, we define the **embedded image** $f(\mathcal{Y})$ of f to be the smallest embedded substack of $\mathcal{X}$ containing $\mathrm{im}(f)$. Embedded image corresponds to the usual notion of image of maps of topological spaces. It is typically bigger than $\mathrm{im}(f)$ (for example, take f to be a surjective non epimorphic map of topological spaces).

When all the stacks involved are topological spaces, the above notions coincides with the original notions in classical topology.

4.3. **The coarse moduli space.** To an stack $\mathcal{X}$ we associate a topological space X_{mod}, called the *coarse moduli space* of $\mathcal{X}$ (or, loosely speaking, the *underlying space* of $\mathcal{X}$). There is a natural map $\pi_{mod}\colon \mathcal{X} \to X_{mod}$ which is universal among maps from $\mathcal{X}$ to topological spaces.

The coarse moduli space X_{mod} is defined as follows. As a set it is equal to $\pi_0(X(*))$, that is, the set of maps from $*$ to $\mathcal{X}$, up to identifications. For any open substack $\mathcal{U} \subseteq \mathcal{X}$, we have a natural inclusion $\mathcal{U}_{mod} \subseteq X_{mod}$. These are defined to be the open sets of X_{mod}.

Example 4.13. Let $X = [R \rightrightarrows X]$ be a groupoid in topological spaces, and let $\mathcal{X} = [X/R]$ be the quotient stack. Then we have a natural homeomorphism $X_{mod} \cong X/R$, where the latter is the (naive) quotient space of the equivalence relation on X induced by R.

Remark 4.14. The coarse moduli space X_{mod} of a stack $\mathcal{X}$ should not be confused with $\pi_0\mathcal{X}$ of Section 3.4 (Definition 3.12). All we can say is that there is a map $\pi_0\mathcal{X} \to X_{mod}$ which induces a bijection on the set of points (i.e. $(\pi_0\mathcal{X})(*) \to X_{mod}(*)$ is a bijection). This map is neither a monomorphism nor an epimorphism in general (also see Corollary 5.4).

Proposition 4.15. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map of stacks.*

- **i**. *The induced map $f_{mod}\colon X_{mod} \to Y_{mod}$ is continuous. If $g\colon \mathcal{X} \to \mathcal{Y}$ is another map that is 2-isomorphic to f, then $f_{mod} = g_{mod}$.*
- **ii**. **Functoriality**. *There is a natural map $\pi_{mod}\colon \mathcal{X} \to X_{mod}$. This map is functorial, in the sense that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{mod}"'] & \mathcal{Y} \arrow[d, "\pi_{mod}"] \\ X_{mod} \arrow[r, "f_{mod}"'] & Y_{mod} \end{tikzcd}$$

- **iii**. **Universal property**. *For any map $f\colon \mathcal{X} \to Y$ to a topological space, there is a unique continuous map $f_{mod}\colon X_{mod} \to Y$ which makes the following diagram commute:*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{mod}"'] & Y \\ X_{mod} \arrow[ur, "f_{mod}"'] & \end{tikzcd}$$

Proof of part (**i**). This is true since the inverse image of an open substack is again an open substack.

Proof of part (**ii**). Let $W \in \mathsf{Top}$ and let $a \in \mathrm{Ob}\,\mathcal{X}(W)$. We abuse notation and denote the corresponding map $W \to \mathcal{X}$ also by a. We define $\pi_{mod}(a) \colon W \to X_{mod}$ to be the map that sends $w \in W$ to $a(w) \in X_{mod}$. In other words, $\pi_{mod}(a)$ is obtained from $a \colon W \to \mathcal{X}$ by passing to the coarse moduli spaces, which is continuous by Part (**i**). Again, by Part (**i**), 2-isomorphic element of $\mathcal{X}(W)$ gives rise to the same elements in $X_{mod}(W)$. The commutativity of the diagram is obvious.

Proof of part (**iii**). Follows from Part (**i**). $\square$

Lemma 4.16. *Let $\mathcal{X}' \subseteq \mathcal{X}$ be an embedded substack. Assume the induced map $X'_{mod} \to X_{mod}$ is a bijection. Then $\mathcal{X}' = \mathcal{X}$.*

Proof. We have to show that for every topological space W we have $\mathcal{X}'(W) = \mathcal{X}(W)$. If this is not the case, we can find a W and a map $f \colon W \to \mathcal{X}$ such that $W' := f^{-1}(\mathcal{X}') \subset W$ is not equal to W. Pick a point $w \in W$ that is not in W'. Then $f(w)$ is a point of $\mathcal{X}$ that is not in $\mathcal{X}'$. This contradicts the fact that $X'_{mod} \to X_{mod}$ is a bijection. $\square$

Proposition 4.17. *Taking inverse image gives a bijection between the subspaces of X_{mod} and embedded substacks of $\mathcal{X}$. The inverse is given by taking embedded image (see Section 4.2).*

Proof. Follows from Lemma 4.16. $\square$

Corollary 4.18. *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a map of stacks. Then $f(\mathcal{Y}) = \pi_{mod}^{-1}\big(f_{mod}(Y_{mod})\big)$, where $f(\mathcal{Y})$ stands for the embedded image of f (Section 4.2).*

5. Gerbes

In this short section, we quickly review gerbes over Top. The following proposition also serves as a reminder of the definition of a gerbe.

Proposition 5.1. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. The following conditions are equivalent:*

- **i**. *For any topological space W and any $y \in \mathcal{Y}(W)$, there is an open covering $\{U_i\}$ of W such that each $y|_{U_i}$ is equivalent (in the groupoid $\mathcal{Y}(U_i)$) to $f(x_i)$, for some $x_i \in \mathcal{X}(U_i)$. For $x, x' \in \mathcal{X}(W)$, and for any isomorphism $\beta \colon f(x) \Rightarrow f(x')$ in $\mathcal{Y}(W)$, there is an open covering $\{U_i\}$ of W such that each restriction $\beta|_{U_i}$ can be lifted to an isomorphism (in the groupoid $\mathcal{X}(U_i)$) $\alpha_i \colon x|_{U_i} \Rightarrow x'|_{U_i}$.*
- **ii**. *The maps $f \colon \mathcal{X} \to \mathcal{Y}$ and $\Delta \colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are epimorphisms.*
- **iii**. *For any map $g \colon \mathcal{Z} \to \mathcal{X}$ of stacks, g is an epimorphism if and only if $f \circ g$ is so.*
- **iv**. *For any map $g \colon Z \to \mathcal{X}$ from a topological space Z to $\mathcal{X}$, g is an epimorphism if and only if $f \circ g$ is so.*

Proof. Standard. $\square$

Definition 5.2. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. We say that $\mathcal{X}$ is a **relative gerbe** over $\mathcal{Y}$ (via f) if the equivalent conditions of Proposition 5.1 are satisfied. When $\mathcal{Y}$ is a topological space, we simply say that $\mathcal{X}$ is a **gerbe**. This is justified by the next proposition.

Proposition 5.3.

i. *Let $\mathcal{X}$ be a stack. Then $\mathcal{X}$ is a gerbe relative to $\pi_0\mathcal{X}$. Conversely, suppose $\mathcal{X}$ is a gerbe relative to a sheaf Y. Then there is a natural isomorphism of sheaves $\pi_0\mathcal{X} \xrightarrow{\sim} Y$.*

ii. *Let $\mathcal{X}$ be a gerbe relative to a topological space Y. Then there is a unique homeomorphism $X_{mod} \cong Y$ (respecting the map coming from $\mathcal{X}$).*

Proof of part (**i**). This is more or less the definition (as in Proposition 5.1.**i**).

Proof of part (**ii**). By Part (**i**), we have $Y = \pi_0\mathcal{X}$, so Y satisfies the universal property of $\pi_0\mathcal{X}$ (the paragraph after Definition 3.12). In particular, Y satisfies the universal property of X_{mod} (Proposition 4.15.**iii**), so it is uniquely homeomorphic to X_{mod}. □

Corollary 5.4. *Let $\mathcal{X}$ be a stack. Then $\mathcal{X}$ is a gerbe if and only if the natural map $\pi_0\mathcal{X} \to X_{mod}$ is an isomorphism.*

Corollary 5.5. *Let $\mathcal{X}$ be a gerbe and let $x\colon W \to \mathcal{X}$ be a W-point in $\mathcal{X}$. Then we have a natural 2-cartesian diagram*

$$\begin{array}{ccc} B_W\mathbb{I}_x & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ W & \longrightarrow & X_{mod} \end{array}$$

Proof. This follows from Corollary 5.4 and Proposition 3.13. □

Corollary 5.6. *Let $\mathcal{X}$ be a gerbe (over X_{mod}), and assume $\pi_{mod}\colon \mathcal{X} \to X_{mod}$ has a section $s\colon X_{mod} \to \mathcal{X}$. Then we have a natural equivalence $\mathcal{X} \cong B_{X_{mod}}\mathbb{I}_s$.*

A gerbe $\mathcal{X}$ is called **trivial** if $\pi_{mod}\colon \mathcal{X} \to X_{mod}$ has a section. By Corollary 5.6, every trivial gerbe is of the form $B_W G$, where W is a topological space and G is a sheaf of groups over W. In general, for any gerbe $\mathcal{X}$, the map $\mathcal{X} \to X_{mod}$ has a section after replacing X_{mod} with an open cover. Therefore, any gerbe $\mathcal{X}$ can be covered by open substacks of the form $B_W G$.

For more examples of gerbes see 19.2.

6. A useful criterion for representability

All stacks are over the site Top.

This section concerns a useful technical lemma for proving representability of maps of stacks (Lemma 6.3). Before proving the lemma, we need some preliminary results.

Lemma 6.1. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks over a base category C. Suppose there exists an epimorphism $\mathcal{X}' \to \mathcal{X}$ such that the base extension $\mathcal{X}' \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{X}'$ is an equivalence. Then f is also an equivalence.*

Proof. To show that f is an equivalence we have to show that, for every $X \in \mathsf{C}$, the induced map $\mathcal{Y}(X) \to \mathcal{X}(X)$ is an equivalence of groupoids. By formal nonsense, it is enough to show that, for every map $p\colon X \to \mathcal{X}$, the base extension of f along p, which itself satisfies the condition of the lemma, is an equivalence. So we are reduced to the case $\mathcal{X} = X$, for some $X \in \mathsf{C}$. Since for every stack $\mathcal{X}'$ there exists an epimorphism $X' \to \mathcal{X}'$ from a sheaf of sets X', we may also assume that $\mathcal{X}' = X'$, for some sheaf of sets X'. So, we end up with a cartesian diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "="'] & \mathcal{Y} \arrow[d, "f"] \\ X' \arrow[r] & X \end{tikzcd}$$

where $X' \to X$ is an epimorphism of sheaves of sets. We have to show that f is an isomorphism, that is, for every $W \in \mathsf{C}$, the groupoid $\mathcal{Y}(W)$ is equivalent to the set $X(W)$ via f.

First, we claim that, for every pair of W-points a and b in $\mathcal{Y}(W)$ such that $f(a) = f(b)$, there is a *unique* isomorphism in $\mathcal{Y}(W)$ between a and b. If a and b are both in the image of X', this follows from the fact the above diagram is cartesian. In the general case, we can replace W by an open covering over which a and b are both in the image of X', so on each of the opens in the covering, we find a unique isomorphism between the restrictions of a and b. Since $\mathcal{Y}$ is a stack, these isomorphisms glue to an isomorphism between a and b over W. This proves the claim.

This claim implies two things at the same time. The first one is that $\mathcal{Y}(W)$ is equivalent to a set, so we might as well assume $\mathcal{Y}$ is a sheaf of sets, and switch the notation from $\mathcal{Y}$ to Y. The second thing is that the map $f\colon Y \to X$ of sheaves of sets is injective.

All that is left to check is that f is an epimorphism (note that we are dealing with *sheaves*). But this is obvious from the commutativity of the above diagram, because if we pre-compose f with $X' \to X$ we get $X' \to X$, which is an epimorphism. $\square$

Corollary 6.2. *Let $\mathcal{X}$ be a stack, and let $\{\mathcal{U}_i\}_{i \in I}$ be a covering of $\mathcal{X}$ by open substacks (Definition 4.7). (Here, by covering we mean that the map $\coprod \mathcal{U}_i \to \mathcal{X}$ is an epimorphism, or, equivalently, the induced map $\coprod \mathcal{U}_{i,mod} \to X_{mod}$ of topological spaces, which is just a union of open embeddings, is surjective.) Assume each $\mathcal{U}_i$ is equivalent to a topological space. Then, so is $\mathcal{X}$.*

Proof. Consider the map of $\pi_{mod}\colon \mathcal{X} \to \mathcal{X}_m$. By assumption, the base extension of this map via the epimorphism $\coprod \mathcal{U}_i^{mod} \to X_{mod}$ is an equivalence of stacks. The result follows from Lemma 6.1. $\square$

Now, we come to the main lemma.

Lemma 6.3. *Consider a 2-cartesian diagram of stacks*

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "a"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Y}' \arrow[r, "b"'] & \mathcal{Y} \end{tikzcd}$$

in which the horizontal arrows are epimorphisms. If f' is representable, then so is f.

Proof. We may assume $\mathcal{Y}$, $\mathcal{X}'$ and $\mathcal{Y}'$ are representable (hence denoting them by Y, X' and Y', respectively). We have to show that $\mathcal{X}$ is representable. First we assume that b is a disjoint union of open embeddings, say $Y' = \coprod V_i \to Y$. Let $X' = \coprod U_i$ be the corresponding decomposition for X'. Then $\coprod U_i \to \mathcal{X}$ is a covering of $\mathcal{X}$ by open substacks, each of which equivalent to a topological space. So, by Corollary 6.2, $\mathcal{X}$ itself is equivalent to a topological space.

For the general case, by what we just proved, we may replace Y with an open covering, so we may assume $b\colon Y' \to Y$ has a section. But in this case the result is obvious. Proof is complete. $\square$

7. Pretopological stacks

In the previous section we looked at stacks over the topological site Top. These are simply categories fibered in groupoids over Top (or, loosely speaking, presheaves of groupoids over Top) which satisfy the descent condition. We saw that we can pinpoint a certain class of morphisms of such stacks (*representable* morphisms) to which we could attribute usual properties of maps of topological spaces.

However, stacks over Top, are still too crude to do topology on. We need to be able to approximate such a stack by an honest topological space so as to, possibly, be able to talk about its topological properties. The next definition is our first approximation of a reasonable notion of a *topological* stack. The full-fledged definition (i.e. that of a *topological stack*) will be given in Section 13.

Definition 7.1. Let $\mathcal{X}$ be a stack over Top. A **chart** for $\mathcal{X}$ is a representable epimorphism $p\colon X \to \mathcal{X}$ from a topological space X to $\mathcal{X}$. If such a chart exists, we say $\mathcal{X}$ is a **pretopological stack**.

First we notice that the condition on representability of the diagonal which appears in standard texts on stacks is implied by the definition.

Proposition 7.2. *Let $\mathcal{X}$ be a pretopological stack. Then, the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.*

Proof. Let $p\colon X \to \mathcal{X}$ be a chart for $\mathcal{X}$. Base extend Δ by $(p,p)\colon X \times X \to \mathcal{X} \times \mathcal{X}$. We obtain the following cartesian diagram:

$$\begin{tikzcd} X \times_{\mathcal{X}} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ X \times X \arrow[r, "(p,p)"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Since $X \times_{\mathcal{X}} X$ is representable and (p,p) is an epimorphism, Lemma 6.3 applies. $\square$

Corollary 7.3. *Let $f\colon X \to \mathcal{X}$ be a map from a topological space to a pretopological stack. Then f is representable.*

Corollary 7.4. *The 2-category of pretopological stacks is closed under 2-fiber product. (Also see Section 9.)*

Proof. Let $\mathcal{Y} \to \mathcal{X}$ and $\mathcal{Z} \to \mathcal{X}$ be morphisms of pretopological stacks. Let $Y \to \mathcal{Y}$ and $Z \to \mathcal{Z}$ be charts. Then it is easy to check that $Y \times_{\mathcal{X}} Z$, which is a topological space by Corollary 7.3, is a chart for $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$. $\square$

Corollary 7.5. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism of pretopological stacks. Then* $\mathrm{im}(f)$ *is pretopological (but not necessarily embedded).*

Proof. Denote $\mathrm{im}(f)$ by $\mathcal{I}$. Let $p\colon X \to \mathcal{X}$ be a chart for X. Then, the induced map $X \to \mathrm{im}(f)$ is an epimorphism; so, $\mathrm{im}(f \circ p) = \mathcal{I}$. Let $R := X \times_{\mathcal{I}} X = X \times_{\mathcal{Y}} X$. Since $\mathcal{Y}$ is pretopological, Corollary 7.3 implies that R is an honest topological space. Therefore, $X \to \mathcal{I}$ is representable, because its base extension along the epimorphism $X \to \mathcal{I}$ is the map $R \to X$, which is representable (Lemma 6.3). $\square$

Corollary 7.6. *Let $\mathcal{X}$ be a pretopological stack. Then the inertia stack $\mathcal{I}_{\mathcal{X}}$ is representable over $\mathcal{X}$ (i.e. the map $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is representable).*

Proof. This follows from Proposition 7.2, because $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is the base extension of the representable map $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$ along $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$. $\square$

Let $\mathcal{X}$ be a pretopological stack, and let $p\colon X \to \mathcal{X}$ be a chart for it. Set $R = X \times_{\mathcal{X}} X$. Since p is representable, R is again a topological space. Thus, we obtain a groupoid in topological spaces $[R \rightrightarrows X]$. By Proposition 3.4, we have an equivalence $[X/R] \to \mathcal{X}$. Conversely, given a groupoid in topological space $[R \rightrightarrows X]$, we can form a quotient stack $\mathcal{X} = [X/R]$. The natural map $P\colon X \to \mathcal{X}$ is then a representable (Lemma 6.3) epimorphism. Therefore, $\mathcal{X}$ is indeed a pretopological stack.

The above correspondence can be made precise by saying that, the 2-category of pretopological stacks equipped with a chart, is equivalent to the 2-category of groupoids in topological spaces. This correspondence gives rise to a correspondence between pretopological stacks (this time without a fixed chart) and *Morita equivalence classes* of topological groupoids. We will say a few words on this in Section 8.

Under the above correspondence, any property of the source (and target) map of a groupoid $[R \rightrightarrows X]$ is reflected as a property of the chart $p\colon X \to [X/R]$ (assuming this property is invariant under base change). For instance, the source and target maps of $[R \rightrightarrows X]$ are open, étale, fibration etc. if and only if the chart $p\colon X \to [X/R]$ is so.

When G is a topological group acting on a topological space (or, more generally, a presheaf of groups acting on a presheaf of sets), we can form the groupoid $[G \times X \rightrightarrows X]$ where the source and target maps are the projection map and the action map, respectively. This groupoid is called the **action groupoid** of this action. The following simple lemma tells us when a topological groupoid is the action groupoid of a discrete group action.

Lemma 7.7. *Let $[R \rightrightarrows X]$ be a topological groupoid. Assume X is connected and R is a disjoint union of components, each of which mapping homeomorphically to X via source and target maps. Then $[R \rightrightarrows X]$ is the action groupoid of a discrete group acting on X.*

Proof. Easy. $\square$

The **stabilizer group** of a groupoid $[R \overset{s,t}{\rightrightarrows} X]$ is defined to be the relative (topological) group $\mathbb{I}_X \to X$, where $\mathbb{I}_X$ is defined by the following cartesian diagram:

$$\begin{array}{ccc} \mathbb{I}_X & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ X & \underset{\Delta}{\longrightarrow} & X \times X \end{array}$$

If $\mathcal{X} = [X/R]$, for some topological groupoid $[R \rightrightarrows X]$, we have the following cartesian diagram

$$\begin{array}{ccc} \mathbb{I}_X & \longrightarrow & \mathcal{I}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

where $\mathcal{I}_{\mathcal{X}}$ is the inertia stack of $\mathcal{X}$ (Section 3.4).

If we denote the chart $X \to [X/R]$ be p, then (the sheaf represented by $\mathbb{I}_X$) can be identified with what we would call $\mathbb{I}_p$ by the terminology of Section 3.4.

The fiber of $\mathbb{I}_X \to X$ over a point x in X is isomorphic, as a topological group, to the inertia group $\mathbb{I}_{p(x)}$ at $p(x)$, so $\mathcal{I}_{\mathcal{X}}$ can be thought of as organizing all the inertia groups in one global family. We will see in Section 17 that an element in a group $\mathbb{I}_x$ can be viewed as a "ghost loop" at x which is not visible if we only look at it in the coarse moduli space X_{mod} of $\mathcal{X}$ (Section 4.3). So $\mathcal{I}_{\mathcal{X}}$ is parameterizes all these ghost loops.

The inertia stack of a topological space is trivial (i.e $\mathcal{I}_{\mathcal{X}} \cong \mathcal{X}$), but the converse is not true (Example 7.8 below). We call a pretopological stack with trivial inertia stack a **quasitopological space** or **quasirepresentable**. A pretopological stack $\mathcal{X}$ is a quasitopological space if and only if for every topological space W, the groupoid of sections $\mathcal{X}(W)$ is equivalent to a set (so $\mathcal{X}$ is really a sheaf of sets). Similarly one can define the notion of a **quasirepresentable map** of stacks.

One can produce examples of pretopological spaces that are not a topological space by considering group actions that are not properly discontinuous. I learned the following example from Angelo Vistoli.

Example 7.8.

1. Let $\mathbb{Q}$, viewed as a discrete group, act on $\mathbb{R}$ by translations. Then $[\mathbb{R}/\mathbb{Q}]$ is a quasitopological space (since the action has no fixed points), but it is not a topological space. An easy way to see this is to compare the fiber products $\mathbb{R} \times_{\mathbb{R}/\mathbb{Q}} \mathbb{R}$ and $\mathbb{R} \times_{[\mathbb{R}/\mathbb{Q}]} \mathbb{R}$. The former is homeomorphic to $\mathbb{R} \times \mathbb{Q}$, where $\mathbb{Q}$ is endowed with its subspace topology from $\mathbb{R}$. But the latter is homeomorphic to $\mathbb{R} \times \mathbb{Q}$, where $\mathbb{Q}$ is endowed with the discrete topology.

2. Consider the previous example, but now assume that $\mathbb{Q}$ is endowed with the subspace topology from $\mathbb{R}$. Again, the quotient stack $[\mathbb{R}/\mathbb{Q}]$ is a quasitopological space, but not a topological space (see the next proposition).

The following proposition tells us exactly when a quotient stack is a topological space.

Proposition 7.9. *Let $[R \rightrightarrows X]$ be a topological groupoid. Then the quotient stack $[X/R]$ is a quasitopological space if and only if the map $R \to X \times X$ is injective. It is a topological space if and only if the following two conditions hold:*

- *$R \to X \times X$ is an embedding;*
- *$X \to X/R$ is an epimorphism.*

In this case, we have $[X/R] \cong X/R$.

8. A few words on Morita equivalence

We saw in the previous section that a topological groupoid gives rise to a pretopological stack by the quotient stack construction. Several topological groupoids may lead to equivalent quotient stacks. Such groupoids are called **Morita equivalent**.

Since the language of groupoids and Morita equivalences is frequently used in the literature, we briefly mention how it fits in the stacky point of view.

All groupoid are assumed to be topological groupoids.

Let $[R' \rightrightarrows X']$ and $[R \rightrightarrows X]$ be groupoids, and let $F\colon [R' \rightrightarrows X'] \to [R \rightrightarrows X]$ be a map of groupoid. By this we mean a pair of maps $F_0\colon X' \to X$ and $F_1\colon R' \to R$ satisfying the obvious conditions. This induces a map of quotient stacks $f\colon [X'/R'] \to [X/R]$. We say f is an *elementary Morita equivalence* if $F_0\colon X' \to X$ is an epimorphism and the induced map $R' \to R|_{X'} = (X' \times X') \times_{X \times X} R$ is a homeomorphism.[6]

It is easy to show that, when F is an elementary Morita equivalence, the induced map $f\colon [X'/R'] \to [X/R]$ of quotient stacks is an equivalence. The converse is not in general true. That is, if $F\colon [R' \rightrightarrows X'] \to [R \rightrightarrows X]$ is a map of groupoids such that the induces map $f\colon [X'/R'] \to [X/R]$ is an equivalence, it does not necessarily follow that F is an elementary Morita equivalence. But we have the following result.

Proposition 8.1. *Let $F\colon [R' \rightrightarrows X'] \to [R \rightrightarrows X]$ be a map of groupoids. Assume in addition that $[R \rightrightarrows X]$ is an étale groupoid (that is, the source and target maps are local homeomorphisms). Then the induced map $f\colon [X'/R'] \to [X/R]$ of quotient stacks is an open embedding if an only if $U = f(X')$ is an open subset of X and the induced map $[R' \rightrightarrows X'] \to [R|_U \rightrightarrows U]$ is an elementary Morita equivalence. Furthermore, f is an equivalence if and only if $\mathcal{O}(U) = X$.*

We will not use this result in this paper and leave it as an exercise to the reader.

Given two groupoids $[R' \rightrightarrows X']$ and $[R \rightrightarrows X]$ whose quotient stack are equivalent, it is not necessarily true that there is map $F\colon [R' \rightrightarrows X'] \to [R \rightrightarrows X]$ inducing the equivalence. All we can say is that, there is another groupoid $[R'' \rightrightarrows X'']$, and a pair of elementary Morita equivalences as in the diagram

$$\begin{tikzcd} & {[R'' \rightrightarrows X'']} \arrow[dl, "\text{Elem. Morita}"'] \arrow[dr, "\text{Elem. Morita}"] & \\ {[R' \rightrightarrows X']} & & {[R \rightrightarrows X]} \end{tikzcd}$$

[6]We can alternatively define elementary Morita equivalence by requiring F_0 to be an open covering. Everything in this section remains valid, except we have to modify the statement of Proposition 8.1.

More generally, given a morphism $f\colon [X'/R'] \to [X/R]$, we can find a groupoid $[R'' \rightrightarrows X'']$, an elementary Morita equivalence $[R'' \rightrightarrows X''] \to [R' \rightrightarrows X']$, and a map of groupoids $F\colon [R'' \rightrightarrows X''] \to [R \rightrightarrows X]$ that induces f after passing to quotient stacks:

$$\begin{tikzcd} & {[R'' \rightrightarrows X'']} \arrow[dl, "\text{Elem. Morita}"'] \arrow[dr, "F"] & \\ {[R' \rightrightarrows X']} & & {[R \rightrightarrows X]} \end{tikzcd}$$

In fact, given a finite collection of maps $f_1, f_2, \cdots, f_n\colon [X'/R'] \to [X/R]$, we can choose $[R'' \rightrightarrows X'']$ so that it works simultaneously for all f_i.

We can also interpret 2-morphisms of the 2-category of pretopological stacks in this way. We will describe briefly how it works, but will not be using it later.

Let $F, G\colon [R' \rightrightarrows X'] \to [R \rightrightarrows X]$ be maps of groupoids and $f, g\colon [X'/R'] \to [X/R]$ the induced maps on the quotient stacks. To describe the 2-morphisms between f and g we make use of the "morphism groupoid" $[P \rightrightarrows R]$ of $[R \rightrightarrows X]$, where P is defined by the following fiber product:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & R \arrow[d, "(s,t)"] \\ R \times R \arrow[r, "(s,s)"'] & X \times X \end{tikzcd}$$

The source map of the groupoid $[P \rightrightarrows R]$ is the top horizontal map of the above square. The target map is given by the composition

$$P = (R \times R) \times_{X \times X} R \xrightarrow{(\iota, id, id)} R \times_X R \times_X R \xrightarrow{\text{mult.}} R$$

where ι is the involution and the last arrow is the multiplication of three composable arrows. We leave it to the reader to figure out the rest of the structure on $[P \rightrightarrows R]$. (*Remark.* This is a special case of the construction of Section 9: take all three groupoids to be $[R \rightrightarrows X]$.)

There are natural maps of groupoids $S, T\colon [P \rightrightarrows R] \to [R \rightrightarrows X]$. It can be shown that, the set of 2-morphisms $f \Rightarrow g$ between $f, g\colon [X'/R'] \to [X/R]$ is in natural bijection with the set of groupoid morphisms $H\colon [R' \rightrightarrows X'] \to [P \rightrightarrows R]$ such that $S \circ H = F$ and $T \circ H = G$.

One can formulate the above discussion as an equivalence between the 2-category of pretopological stacks and the 2-category of groupoids with elementary Morita equivalences "inverted". We will not need it here, so we will leave it as a challenge for the reader!

9. Fiber products

Let C be a Grothendieck site. As we saw in Section 3.1, the 2-category of categories fibered in groupoids over C has 2-fiber products (sometimes called *homotopy* *fiber products*). The 2-subcategory St_{C} of stacks is closed under this 2-fiber product. This is due to the fact that stackification commutes with 2-fiber products.

Now, let $\mathsf{C} = \mathsf{Top}$ be the site topological spaces. We saw in Section 7 that the 2-subgategory $\mathsf{PretopSt}_{\mathsf{Top}}$ of pretopological stacks is also closed under 2-fiber products (Corollary 7.4). Here we present another proof that can also be used later on to show that the 2-subcategories of topological stacks, and Deligne-Mumford topological stack (as defined in the subsequent sections) are also closed under 2-fiber products.

Assume we are given a diagram of pretopological stacks

$$\begin{array}{ccc} & & \mathcal{Y} \\ & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow[g]{} & \mathcal{X} \end{array}$$

As we saw in Section 8, we can choose groupoid presentations for $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, and also for the maps between them. So we may assume that the above diagram comes from the following diagram of topological groupoids:

$$\begin{array}{ccc} & & [S \rightrightarrows Y] \\ & & \downarrow{\scriptstyle F} \\ [T \rightrightarrows Z] & \xrightarrow[G]{} & [R \rightrightarrows X] \end{array}$$

We define the *fiber product*[7] of this diagram to be the topological groupoid whose base (i.e. the“object space”) is

$$P_0 := (Y \times Z) \times_{X \times X} R \quad (\text{using} \quad Y \times Z \xrightarrow{(F_0, G_0)} X \times X.)$$

and whose“arrow space” is

$$P_1 := (S \times T) \times_{X \times X} R \quad (\text{using} \quad S \times T \xrightarrow{(F_0 \circ s, G_0 \circ s)} X \times X).$$

The source map of the groupoid $[P_1 \rightrightarrows P_0]$ is just the base extension of the map

$$S \times T \xrightarrow{(s,s)} Y \times Z,$$

so it will be nicely behaved (e.g. smooth, étale, LF etc.) provided those of $[S \rightrightarrows Y]$ and $[T \rightrightarrows Z]$ are. The involution $\iota_P \colon P_1 \to P_1$ is defined by

$$(S \times T) \times_{X \times X} R \xrightarrow{(\iota_S, \iota_T, \varphi)} (S \times T) \times_{X \times X} R$$

where ι stands for the involution, and the map φ is defined by the composition

$$(S \times T) \times_{X \times X} R \xrightarrow{(F_1 \circ \iota_S, G_1, \mathrm{id})} R \times_X R \times_X R \xrightarrow{\text{mult.}} R$$

[7]Despite its messy appearance, the construction is quite simple. It is simply the construction of homotopy fiber product of (discrete) groupoids translated in the categorical language.

Having defined ι_P, we define the target map $t\colon P_1 \to P_0$ to be the composition $s \circ \iota_P$. We leave it to the reader to figure out the multiplication map.

It is immediate from the construction that the following diagram of presheaves of groupoids is 2-cartesian (see Section 3.2 for notation):

$$\begin{array}{ccc} \lfloor P_0/P_1 \rfloor & \longrightarrow & \lfloor Y/S \rfloor \\ \downarrow & & \downarrow \\ \lfloor Z/T \rfloor & \longrightarrow & \lfloor X/R \rfloor \end{array}$$

Since stackification commutes with 2-fiber products, the above diagram remains 2-cartesian after stackification. Therefore, $\mathcal{P} := [P_0/P_1]$ is naturally equivalent to the 2-fiber $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$.

Remark 9.1. Of course, there is nothing specific about Top; the same construction is valid for algebraic or analytic stacks.

10. Inertia groups and residue gerbes of pretopological stacks

When $\mathcal{X}$ is a pretopological stack, for every point $x \in \mathcal{X}$ the inertia group is naturally a topological group. When viewed as a topological group we denote it by $\mathbb{I}_x$, and when viewed as discrete group we denote it by I_x.

The residue stack Γ_x at a point x also has a nice description, as seen in the Proposition 10.1 below. In particular, it is a gerbe. Whenever we deal with $*$-valued points, we switch the terminology from residue stacks to **residue gerbe**s. The residue gerbe Γ_x of a point x is of very local nature, in the sense that, for any substack (not necessarily open or closed) $\mathcal{Y}$ of $\mathcal{X}$ which contains x (i.e., $x\colon * \to \mathcal{X}$ factors through $\mathcal{Y}$), the residue gerbe of x as a point in $\mathcal{Y}$ is the same as the residue gerbe of x viewed as a point in $\mathcal{X}$. In fact, the same is true for the residue stack of any W-valued point.

Proposition 10.1. *Let $\mathcal{X}$ be a pretopological stack and let x be a point in $\mathcal{X}$. Then we have a natural equivalence $\Gamma_x \cong \mathcal{B}\mathbb{I}_x$.*

Proof. The result follows from Corollary 3.17 with $W = *$. □

Consider the moduli map $\pi_{mod}\colon \mathcal{X} \to X_{mod}$. The residue gerbe Γ_x maps to the point $\bar{x} := \pi_{mod}(x) \in X_{mod}$. It may appear that Γ_x is the fiber of π_{mod} over $\bar{x}$. It turns out, however, that this is not always the case. The next proposition tells us exactly when this is the case.

Proposition 10.2. *Let $\mathcal{X}$ be a pretopological stack, and let $p\colon X \to \mathcal{X}$ be a chart for it. Denote the corresponding groupoid by $[R \rightrightarrows X]$. Let x be a point in $\mathcal{X}$. Pick a lift $x' \in X$ of x (so $p(x')$ is equivalent to x), and let $X' = \mathcal{O}(x')$ be its orbit (with the subspace topology). Denote the restriction of $[R \rightrightarrows X]$ to X' by $[R' \rightrightarrows X']$. Finally, let $\Gamma'_x = \mathrm{im}(X')$ (via the natural map $X' \to \mathcal{X}$).*

i. *There are 2-cartesian diagrams:*

$$\begin{array}{ccc} \Gamma_x & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \xrightarrow[\bar{x}]{} & \pi_0\mathcal{X} \end{array} \qquad\qquad \begin{array}{ccc} \Gamma'_x & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \xrightarrow[\bar{x}]{} & X_{mod} \end{array}$$

ii. *There is a natural equivalence* $[X'/R'] \xrightarrow{\sim} \Gamma'_x$.

iii. *There is a natural equivalence* $\mathcal{B}\mathbb{I}_x \xrightarrow{\sim} \Gamma_x$.

iv. Γ_x *is a substack of* Γ'_x. *Indeed,* Γ'_x *is the smallest embedded substack of* $\mathcal{X}$ *containing* Γ_x *(i.e.* Γ'_x *is the embedded image of* $x\colon * \to \mathcal{X}$*).*

v. $\Gamma_x = \Gamma'_x$ *if and only if* $t\colon s^{-1}(x') \to X'$ *is an epimorphism. In this case, we have a 2-cartesian diagram*

$$\begin{array}{ccc} \mathcal{B}\mathbb{I}_x & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & X_{mod} \end{array}$$

Proof of part (**i**). The left square is 2-cartesian by Corollary 3.16. To prove that the right square is 2-cartesian, let Γ be the substack of $\mathcal{X}$ that is equivalent to the 2-fiber product $* \times_{X_{mod}} \mathcal{X}$; we have to show that $\Gamma = \Gamma'_x$. There is a natural map $X' \to \Gamma$ which makes Γ'_x a substack of Γ. It is enough to show that this map is an epimorphism. This follows from the fact that the following diagram is 2-cartesian:

$$\begin{array}{ccc} X' & \rightarrow & X \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \mathcal{X} \end{array}$$

Proof of part (**ii**). This follows from Proposition 3.11.

Proof of part (**iii**). This is Proposition 10.1.

Proof of part (**iv**). Since $x\colon * \to \mathcal{X}$ factors through Γ'_x, the inclusion $\Gamma_x \subseteq \Gamma'_x$ is obvious. The second statement follows from Corollary 4.18.

Proof of part (**v**). The equality $\Gamma_x = \Gamma'_x$ holds if and only if $x\colon * \to \Gamma'_x$ is an epimorphism. Using the cartesian diagram

$$\begin{array}{ccccc} s^{-1}(x') & \xrightarrow{t} & X' & \rightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \Gamma'_x & \rightarrow & \mathcal{X} \end{array}$$

(in which the vertical maps are epimorphisms), the bottom left map being an epimorphism is equivalent to the top left map being an epimorphism.

The last statement follows from (**i**) and (**iii**). $\square$

The point of the above proposition is that, given a *transitive* groupoid $[R \rightrightarrows X]$, the quotient stack $[X/R]$ is not necessarily equivalent to $\mathcal{B}\mathbb{I}$, where $\mathbb{I}$ is the stabilizer group (viewed as a topological group) of a point on X. The former is typically larger than the latter. The equality holds, if and only if the assumption (**v**) of the proposition is satisfied. When (**v**) is satisfied at every point on a pretopological stack $\mathcal{X}$, we can think of $\mathcal{X}$ as a "family of classifying stacks $\mathcal{B}\mathbb{I}_x$, parameterized by X_{mod}".

Example 10.3. Let $[R \rightrightarrows X]$ be a groupoid and assume $x \in X$ is a point for which the orbit $\mathcal{O}(x)$ is discrete. Then the condition (**v**) of the proposition of the proposition is satisfied at x. So we have a 2-cartesian diagram

$$\begin{tikzcd} \mathcal{B}\mathbb{I}_x \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ * \arrow[r] & X_{mod} \end{tikzcd}$$

There are two major classes of stacks for which the condition (**v**) of Proposition 10.2 is automatically satisfied at every point: Deligne-Mumford topological stacks (Section 14) and pretopological gerbes (use Corollary 5.4 and Proposition 10.2.**i**).

Example 10.4. Let $X = \mathbb{Q}$ with the subspace topology induced from $\mathbb{R}$. Let $\mathbb{Q}$ (viewed as a discrete group) act on X by translations, and let $\mathcal{X}$ be the quotient stack. Since this action is transitive, we have $X_{mod} = *$. So, $\mathcal{X}$ is a quasitopological space that is not a topological space.

Example 10.5. Another counter-example is obtained by taking X to be the trivial topology on a set with 2 elements and letting $\mathbf{Z}_2$ act on it by swapping the elements. The coarse moduli space of the quotient stack is obviously $*$. We have a natural representable map $* \to [X/\mathbf{Z}_2]$ that identifies $*$ with a substack of $[X/\mathbf{Z}_2]$, but this substack is not embedded.

11. The coarse moduli space in the presence of open charts

In this section we assume all out pretopological stacks admit open charts.

In the presence of an *open* chart for a pretopological stack $\mathcal{X}$, the coarse moduli space is better-behaved. We will address this issue in this section. Recall that, by an open chart $p\colon X \to \mathcal{X}$ for $\mathcal{X}$ we mean one for which p is an open map (Definition 4.5 and Example 4.6).

We can extend the notion of an open map to morphisms of pretopological stacks that are not necessarily representable.

Definition 11.1. We say that a (not necessarily representable) map $f\colon \mathcal{Y} \to \mathcal{X}$ of (not necessarily pretopological) stacks is **open** if there exists diagram of stacks

$$\begin{tikzcd} \mathcal{Z} \arrow[d, "h"'] \arrow[dr, "g"] & \\ \mathcal{Y} \arrow[r, "f"'] & \mathcal{X} \end{tikzcd}$$

where h is a surjective map and g is a representable open map. (For a not necessarily representable map h surjective means h_{mod} is surjective.)

It is easy to show that base extension of an open map is open, and so is the composition of two open maps. It is also easy to show that when f is representable this definition coincides with the previous definition.

Lemma 11.2. *Consider a diagram of stacks (not necessarily pretopological)*

$$\begin{tikzcd} \mathcal{Z} \arrow[d, "h"'] \arrow[dr, "g"] & \\ \mathcal{Y} \arrow[r, "f"'] & \mathcal{X} \end{tikzcd}$$

in which h is surjective and g is open. Then f is open.

Proof. Since g is open, there is a stack $\mathcal{Z}'$ and maps $h' \colon \mathcal{Z}' \to \mathcal{Z}$ and $g' \colon \mathcal{Z}' \to \mathcal{X}$ such that h' is surjective and g' is representable and open. The following diagram does the job:

$$\begin{tikzcd} \mathcal{Z}' \arrow[d, "h \circ h'"'] \arrow[dr, "g'"] & \\ \mathcal{Y} \arrow[r, "f"'] & \mathcal{X} \end{tikzcd}$$

□

Exercise. Show that if $f \colon \mathcal{Y} \to \mathcal{X}$ is open, then so is $f_{mod} \colon Y_{mod} \to X_{mod}$, but the converse is not necessarily true.

Lemma 11.3. *Let $\mathcal{X}$ be a pretopological stack, and $[R \rightrightarrows X]$ a groupoid presentation for it.*

- **i**. *There is a natural homeomorphism $X/R \xrightarrow{\sim} X_{mod}$, where X/R is the (naive) quotient space of the equivalence relation induced on X from R.*
- **ii**. *If the source map of the groupoid is open, then so is $X \to X_{mod} = X/R$.*

In particular, if $\mathcal{X}$ is a pretopological stack that admits an open chart, then the moduli map $\pi_{mod} \colon \mathcal{X} \to X_{mod}$ is open.

Proof. Easy. □

If for a stack $\mathcal{X}$ the moduli map $\pi_{mod} \colon \mathcal{X} \to X_{mod}$ is open, then it is uniquely characterized by the following three properties: open, continuous, bijection. By the above proposition, this is the case when $\mathcal{X}$ is the quotient stack of a topological groupoid whose source (hence also target) map is open.

Corollary 11.4 (Invariance under base change). *Let $\mathcal{X}$ be a stack such that the moduli map $\pi_{mod} \colon \mathcal{X} \to X_{mod}$ is open (e.g. a pretopological stack that admits an open chart), and let $f \colon Y \to X_{mod}$ be a continuous map. Set $\mathcal{Y} = Y \times_{X_{mod}} \mathcal{X}$. Then, the projection map $\pi \colon \mathcal{Y} \to Y$ makes Y into the coarse moduli space of $\mathcal{Y}$.*

Proof. It is easily checked that $\pi \colon \mathcal{Y} \to Y$ induces a natural continuous bijection $Y_{mod} \to Y$. The map $\pi \colon \mathcal{Y} \to Y$ is open, being base extension of an open map. This implies that $Y_{mod} \to Y$ is also open (Lemma 11.2), hence a homeomorphism. □

12. Quotient stacks as classifying spaces for torsors

Let $[R \rightrightarrows X]$ be a groupoid in Top. There is an alternative description of the quotient stack $[X/R]$ in terms of torsors which is both technically and conceptually very important. Roughly speaking, the quotient stack $[X/R]$ *parameterizes* torsors for the groupoid $[R \rightrightarrows X]$ (see Definition 12.3). For example, if $[G \rightrightarrows *]$ is the action groupoid of a topological group G acting (trivially) on a point, then maps from a topological space Y into $[*/G]$ correspond to G-torsors over Y. In this sense, $\mathcal{B}G = [*/G]$ can be thought of as the *classifying space* of G.

Definition 12.1. A map $[R \rightrightarrows X] \to [R' \rightrightarrows X']$ of groupoids is called **cartesian** if the following square is cartesian

$$\begin{tikzcd} R \arrow[r] \arrow[d, "s"'] & R' \arrow[d, "s"] \\ X \arrow[r] & X' \end{tikzcd}$$

Note that the same will be true for t.

Remark 12.2. The above definition is equivalent to saying that the following diagram of presheaves of groupoids is 2-cartesian:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ \lfloor X/R \rfloor \arrow[r] & \lfloor X'/R' \rfloor \end{tikzcd}$$

Since stackification commutes with fiber products, this diagram remains 2-cartesian after stackification.

As far as I know, the following definition is due to Kai Behrend.

Definition 12.3. Let $[R \rightrightarrows X]$ be a groupoid in Top, and let W be a topological space. By an R-**torsor** (more precisely, a $[R \rightrightarrows X]$-torsor) over W we mean an epimorphism $T \to W$ of topological spaces, together with a cartesian map of groupoids

$$[T \times_W T \rightrightarrows T] \to [R \rightrightarrows X].$$

Given R-torsors $T \to W$ and $T' \to W'$, we define a *morphism* of R-torsors from T to T' to be a cartesian square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T' \arrow[d] \\ W \arrow[r] & W' \end{tikzcd}$$

such that the induced diagram of groupoids commutes:

$$\begin{tikzcd}[T \times_W T \rightrightarrows T] \arrow[rr] \arrow[dr] & & [T' \times_{W'} T' \rightrightarrows T'] \arrow[dl] \\ & [R \rightrightarrows X] & \end{tikzcd}$$

The category of all R-torsors, over various W, is fibered in groupoids over Top via the forgetful functor (that out of everything only remembers W).

Example 12.4.

1. Take $[R \rightrightarrows X]$ be the action groupoid $[G \times X \rightrightarrows X]$ of a topological group G acting on a topological space X. An R-torsor over W consists of a pair (T, α), where $T \to W$ is a G-torsor (that is, a locally trivial principal G-space over W), and $\alpha\colon T \to X$ is a G-equivariant map. For a fixed W, a morphism between R-torsors (T, α) and (T', α') over W is a G-equivariant map $a\colon T \to T'$ relative to W such that the following triangle commutes:

$$\begin{tikzcd} T \arrow[rr, "a"] \arrow[dr, "\alpha'"'] & & T' \arrow[dl, "\alpha"] \\ & X & \end{tikzcd}$$

2. In the previous example, assume the action of G on X is trivial. Then, an R-torsor over W is of a pair (T, β) consisting of a G-torsor T over W and continuous map $\beta\colon W \to X$. When $X = *$ an R-torsor is simply a G-torsor.
3. We could also consider the previous examples with everything being relative to a base B. So, X is now a space over B, and G is a topological group over B, acting on X (relative to B). A particular case of interest is when $X = B$.

We say an R-torsor $T \to W$ is **trivial** if it admits a section $\sigma\colon W \to T$. Having fixed such a section, the set of R-torsor morphisms $a\colon T \to T'$ to an R-torsor $T' \to W$ is in natural bijection with the set of sections $\sigma'\colon W \to T'$ for which the following square commutes:

$$\begin{tikzcd} T' \arrow[r] & X \\ W \arrow[u, dotted, "\sigma'"] \arrow[r, "\sigma"'] & T \arrow[u] \end{tikzcd}$$

The bijection is given by $a \mapsto a(\sigma)$.

Remark 12.5. An arbitrary R-torsor T over W is locally trivial. That means, there is an open covering $\{U_i\}$ of W such that the restriction of T to each U_i is trivial. This is because $T \to W$ is an epimorphism.

The following proposition explains why R-torsors are related to stacks.

Theorem 12.6. *Let $[R \rightrightarrows X]$ be a groupoid in* Top*. We have a natural equivalence (of categories fibered in groupoids over* **Top***)*

$$[X/R] \xrightarrow{\sim} \mathrm{Tors}_R\,.$$

In particular, Tors_R *is a pretopological stack over* Top*.*

Proof. Denote $[X/R]$ by $\mathcal{X}$. We define a functor $\Theta\colon \mathcal{X} \to \mathrm{Tors}_R$ as follows. Let $W \in \mathsf{Top}$ and pick on object $f \in \mathcal{X}(W)$. We denote the corresponding map $W \to \mathcal{X}$ also by f. We define $\Theta(f)$ to be the R-torsor $T \to W$, where $T = W \times_{\mathcal{X}} X$ (the map from W to $\mathcal{X}$ being f). The fact that $T \to W$ has an R-torsor structure follows from formal properties of 2-fiber products. Similarly, given a morphism $\alpha\colon f \Rightarrow f'$ in $\mathcal{X}(W)$, we obtain an induced map of fiber products $a\colon T \to T'$ (where $T' = W \times_{\mathcal{X}} X$, the map from W to $\mathcal{X}$ being f'). Again, it follows from formal properties of 2-fiber products that this is indeed a map of R-torsors. This defines the functor $\Theta\colon \mathcal{X} \to \mathrm{Tors}_R$.

We now show that Θ is an equivalence. We have to show that Θ is fully faithful and essentially surjective.

Proof of faithfulness. Let $f, f' \in \mathcal{X}(W)$ be W-points (we use the same notation for the corresponding maps $W \to \mathcal{X}$) and $\alpha, \beta\colon f \Rightarrow f'$ morphisms between them in $\mathcal{X}(W)$ (which we view as 2-isomorphisms between the corresponding maps $W \to \mathcal{X}$). Let $T, T' \to W$ be the corresponding R-torsors, with $a, b\colon T \to T'$ the R-torsors morphisms associated to α and β. We have to show that $a = b$ implies $\alpha = \beta$. Since $\mathcal{X}$ is a stack, it is enough to check the equality $\alpha = \beta$ locally, so we may replace W by an open cover and assume that f lifts to X. This is equivalent to saying that the torsors T is trivial. We fix a choice of a lift for f (by this we mean a pair $(\tilde{f}, \gamma)$, where $\tilde{f}\colon W \to X$ is a map of topological spaces and $\gamma\colon f \Rightarrow p \circ \tilde{f}$ is an identification). By definition of 2-fiber product, there is a natural bijection between such lifts and sections of $T \to W$. So, $(\tilde{f}, \gamma)$ corresponds to a section $\sigma\colon W \to T$.

The sections $a(\sigma), b(\sigma)\colon W \to T'$ correspond to the lifts $(\tilde{f}, \alpha^{-1}\gamma)$ and $(\tilde{f}, \beta^{-1}\gamma)$ of f'. If $a = b$, then these two lifts are equal, so $\alpha^{-1}\gamma = \beta^{-1}\gamma$. This implies $\alpha = \beta$.

In fact, the above argument proves the fullness as well, provided we assume the existence of lifts.

Proof of fullness. Let W, f,f', T and T' be as in the previous part. Let $a\colon T \to T'$ be an R-torsor morphism. We have to show that it comes from an $\alpha\colon f \Rightarrow f'$. We saw in the previous part that the statement is true if T is trivial (which implies that T' is also trivial). In the general case, choose a covering $\{U_i\}$ of W over which T becomes trivial (Remark 12.5). Then, we obtain a unique family $\alpha_i\colon f|_{U_i} \Rightarrow f'_{U_i}$ inducing $a|_{U_i}\colon T|_{U_i} \to T'|_{U_i}$. Since the $a|_{U_i}$ are compatible over the double intersections, so will be the α_i (we are using faithfulness). So α_i glue to a global $\alpha\colon f \Rightarrow f'$ over W. Let $b\colon T \to T'$ be the R-torsor map induced by α. Then a and b are equal over every open U_i. It is easy to check that this implies that $a = b$. In other words, a is induced from α.

Proof of essential surjectivity. Let $T \to W$ be an R-torsor over W. By definition, we have a cartesian diagram

$$\begin{tikzcd} T \times_W T \arrow[r] \arrow[d, "pr_1"'] & R \arrow[d] \\ T \arrow[r] & X \end{tikzcd}$$

By Remark 12.2, and using the natural equivalence $[T/T \times_W T] \xrightarrow{\sim} W$, we obtain a 2-cartesian diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & {[X/R]} \end{tikzcd}$$

Denoting the bottom map by $f \colon W \to [X/R]$, we see that the R-torsor $T \to W$ is isomorphic to $\Theta(f)$. This proves the essential surjectivity. $\square$

The upshot is that the quotient stack $[X/R]$ can be thought of as "the moduli stack of R-torsors", with $X \to [X/R]$ being the "universal" R-torsor over it. In other words, an R-torsor T over W gives rise to a map $f \colon W \to [X/R]$ and T is just the pull back of of the universal R-torsor along f. Given two R-torsors T and T' over R, a morphism between them corresponds to a 2-isomorphism between the corresponding classifying maps $f, f' \colon W \to [X/R]$.

The reader is advised to work out the meaning of the above proposition for the special cases considered in Example 12.4.

Part 2. Topological stacks

13. Topological stacks

In this section, we define our main objects of interest, the *topological stacks*. A topological stack is a pretopological stack which admits a chart that is a *local fibration* (LF, for short), where a local fibration is defined to be a member of a class **LF** of continuous maps satisfying certain axioms (see Subsection 13.1 below).

Therefore, our notion of topological stack is not an absolute one and depends on the choice of the class **LF**. The freedom in choosing our own notion of local fibration gives us flexibility in crafting the corresponding notion of topological stack meeting our particular needs. For example, if we need to do homotopy theory with our stacks, *local Serre fibrations* (see Example 13.1) is a good choice. Or, if we want the coarse moduli spaces to be well-behaved we require that all the maps in **LF** to be open.

We will see that the desired properties of a category of topological stacks are often reflected as some axioms imposed on **LF**.

Topological stacks are the topological counterparts of algebraic stacks. In fact, we will see in Section 20 how to associate a topological stack to an algebraic stack (of finite type) over $\mathbb{C}$, much the same way that one associates a topological space to a scheme (of finite type) over $\mathbb{C}$. In Section 14, we consider (weak) Deligne-Mumford topological stack, which are the topological counterparts of algebraic Deligne-Mumford stacks.

This section has two subsections. In the first one we introduce local fibrations and supply some examples. In the second subsection, we use the notion of local fibration to define topological stacks.

13.1. **Local fibrations.** For a pretopological stack to behave nicely, we need to require some kind of 'fibrancy' condition on the chart $p\colon X \to \mathcal{X}$. The stronger the fibrancy condition is, the more manageable our stack becomes. What this exactly means is discussed in more details in Section 16. Let us just mention that, the fibrancy conditions on charts will result in having more freedom in performing push-outs in the corresponding category of stacks.

By a class **LF** of **local fibrations** we mean a collection of continuous maps satisfying the following axioms:

- **LF1**. Every open embedding is a local fibration;
- **LF2**. Local fibrations are closed under composition;
- **LF3**. Being a local fibration is stable under base change and local on the target (Definition 4.4);
- **LF4**. If $f_i\colon X_i \to Y$ is a family of local fibrations, then $\coprod f_i\colon \coprod X_i \to Y$ is also a local fibration.

Having fixed **LF**, we refer to a map in **LF** by an *LF map*.

Note that, by virtue of **LF3**, we can talk about LF maps of stacks. More precisely, we say a representable map $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is LF, if there is a chart $Y \to \mathcal{Y}$ such the base extension of f over Y is LF.

Example 13.1. Let **Fib** be any of the following classes of maps:

1. *Serre fibrations.* These are maps that have the homotopy lifting property for finite CW complexes.
2. *Hurewic fibrations.* These are maps that have the homotopy lifting property for all topological spaces.
3. *Cartesian maps.* These are projection maps of products, i.e $pr_2 \colon X \times B \to B$.
3′. There are variations of the notion of cartesian map in which the fibers are assumed to belong to a certain family of spaces. For instance, we can require the fiber to be a Euclidean space (i.e. homeomorphic to some $\mathbb{R}^n$).
4. *Right invertible maps.* These are maps that admit sections.
5. *Open maps.*
6. *Homeomorphisms.*

Define the corresponding class **LF** of local fibrations by declaring a continuous map $f \colon X \to Y$ of topological spaces to be LF, if for every point $x \in X$ there are open sets $U \subseteq X$ containing x and $V \subseteq Y$ containing $f(x)$ such that $f|_V \colon U \to V$ is in **Fib**. It is easy to check that the resulting class of maps satisfies the axioms for local fibrations.

The local fibrations for Example (**3**) are sometimes called *topological submersion.* Every submersion of differentiable manifolds is a local fibration for Example (**3′**) (Implicit Function Theorem). Local fibrations of Example (**4**) are the epimorphisms. Local fibrations of Example (**5**) are the open maps. Local fibrations of Example (**6**) are the local homeomorphisms.

Example 13.2. Intersection of any two classes of local fibrations is again a class of local fibrations. So we could from new classes such as *open epimorphic local Serre fibrations* and so on.

Every notion of fibration comes with the corresponding notion of **cofibration** (rather, *trivial* cofibration), which is defined using certain lifting properties. So, with *local* fibrations we have the corresponding notion of *local* trivial cofibrations.

Definition 13.3. Let $f \colon X \to Y$ be a map of topological spaces. We say a map $i \colon A \to B$ of topological spaces has *local left lifting property* with respect to f if in every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

and for every point $a \in A$, there exist small enough neighborhoods $U \subseteq B$ of $i(a)$ such that, after replacing i by $i|_U \colon i^{-1}(U) \to U$, the dotted arrow can be filled. For a fixed class **LF** of local fibrations, we say a map $i \colon A \to B$ is a **local (trivial) cofibration** (LTC for short), if it satisfies the local left lifting property with respect to every $f \in$ **LF**.

Lemma 13.4. *Fix a class* **LF** *of local fibrations.*

i. *Every open embedding is LTC.*
ii. *LTC maps are closed under composition.*

iii. *LTC maps are stable under push out along embeddings. That is, if in the following push out diagram i is LTC and j is an embedding, then i' is also LTC:*

$$\begin{tikzcd} A \arrow[r, hook, "j"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r, hook] & B' \end{tikzcd}$$

iv. *Let $i\colon A \to B$ be LTC, and let $U \subseteq B$ be an open set. Then $i|_U : i^{-1}(U) \to U$ is LTC.*

v. *Being LTC is local on the target, that is, if $i\colon A \to B$ is a map of topological spaces such that, for some collection $\{U_i\}$ of open subsets of B whose union contains $i(A)$, the maps $i^{-1}(U_i) \to U_i$ are LTC, then so is i.*

vi. *Let $i\colon A \to B$ be a map of topological spaces. Let $p\colon B' \to B$ be an epimorphism, and let $A' = B' \times_B A$. If $i' : A' \to B'$ is LTC then so is i.*

vii. *Assume $i\colon A \coprod A' \to B$ is LTC. Then $\overline{i(A)} \cap i(A') = \emptyset$.*

Proof. Parts (**i**)-(**v**) are straightforward.

Proof of part (**vi**). Since being LTC is local on the target, we may assume that p has a section $s\colon B \to B'$. Consider a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted, "q"] & Y \end{tikzcd}$$

where $X \to Y$ is LF. Pre-composing the horizontal arrows with p, we obtain the following diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] & X \arrow[d, "f"] \\ B' \arrow[r] \arrow[ur, dotted, "q'"] & Y \end{tikzcd}$$

Since i' is LTC, there is a family $\{U'_i\}$ of open subsets of B' whose union contains $i(A')$ and over which the lift q' exists. Set $U_i := s^{-1}(U'_i)$. Then $\{U_i\}$ is a family of open subsets of B whose union contains $i(A)$ and over which the lift q exists: take $q = q' \circ s$.

Proof of part (**vii**). Consider the local fibration $f = \mathrm{id} \coprod \mathrm{id}\colon B \coprod B \to B$. Let $g = i|_A \coprod i|_{A'} : A \coprod A' \to B \coprod B$. The local left lifting property for the following diagram implies the claim:

$$\begin{tikzcd} A\coprod A' \arrow[r, "g"] \arrow[d, "j"'] & B\coprod B \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted] & B \end{tikzcd}$$

□

Remark 13.5. By part (**vi**) of the above proposition, we can define LTC for every representable map of stacks.

Remark 13.6. For a given class **C** of maps in **Top** satisfying property (**vii**) of Lemma 13.4, let **LF** be the class of all maps $f\colon X \to Y$ with the property that every $i \in \mathbf{C}$ satisfies the local left lifting property with respect to f. It can be checked that **LF** satisfies the axioms **LF1-LF4**. This gives an alternative way to produce classes of local fibrations.

Let us look at some examples of LTC maps, corresponding to classes of local fibrations considered in Example 13.1.

Example 13.7. Numbering corresponds to the numbering in Example 13.1.

- **1**. Every CW inclusion of finite CW complexes is LTC. To prove this, use the fact that every such inclusion is locally homotopy equivalent to the $t = 0$ inclusion $X \hookrightarrow I \times X$, where X is a finite CW complex.
- **2**. Every $i\colon A \to B$ that is a Hurewic cofibration (or only locally on B so) is LTC. This follows from [St], Lemma 4.
- **3**. Every embedding that is a local retract is LTC.
- **3′**. *Locally cartesian maps with Euclidean fibers.* Every embedding that locally satisfies the Tietze extension property is LTC. For instance, every inclusion of a locally-closed subspace in a normal topological space is LTC.
- **4**. The only LTC maps are open embeddings.
- **6**. Every embedding is LTC.

13.2. **Topological stacks.** Fix your favorite class **LF** of local fibrations (see the previous subsection).

Definition 13.8. Let $\mathcal{X}$ be a pretopological stack. We say $\mathcal{X}$ is **topological** if there is a chart $p\colon X \to \mathcal{X}$ that is LF.

For a choice **LF** of a class of local fibrations, we have the corresponding theory of topological stacks. The smaller the class of local fibrations, the smaller the corresponding category of topological stacks, the more flexibility in performing push-outs in the corresponding 2-category of topological stacks (as we will see in Section 16).

Our favorite choices for **LF** are the ones listed in Example 13.1, or their combinations (Example 13.2). After Example 4, which gives us the good old pretopological stacks, the topological stacks of Example 1 give us the most general theory. The ones for Example 3′ appear as underlying topological stacks of differentiable and also algebraic stacks. Topological stack of Example 5 are the ones studied in Section 11. Topological stacks of Example 6 are what we call *weak Deligne-Mumford stacks*. These are studied in the next Section.

Topological stacks form a full 2-subcategory of $\mathsf{St}_{\mathsf{Top}}$, which we denote by TopSt. The Yoneda functor identifies Top with a full 2-subcategory of TopSt.

Proposition 13.9. *Let $\mathcal{X}$ be a pretopological stack.*

- **i**. *If $\mathcal{X}$ is topological, then so is every embedded substack of $\mathcal{X}$.*
- **ii**. *Let $\{\mathcal{U}_i\}_{i\in I}$ be a covering of $\mathcal{X}$ by open substacks. If every $\mathcal{U}_i$ is topological, then so is $\mathcal{X}$.*
- **iii**. *The 2-category* TopSt *is closed under 2-fiber products.*

Proof. Easy. □

There is an inherent difficulty in performing push-outs in the 2-category of topological stacks. In fact, the notion of *local fibration* is precisely designed to bring this issue under control.

An unfortunate fact is that, even simple push-outs of topological *spaces* may no longer be push-outs when viewed in TopSt. It is also unreasonable to expect to be able to quotient out a topological stack modulo a, say, closed substack. For this we *need* to impose certain conditions on the closed substack, or else the quotient can be shown not to exists in any reasonable sense.[8]

Similar problems occur when we try to glue topological stacks along substacks. In Section 16 we will discuss the issue of push-outs in TopSt in more detail.

14. Deligne-Mumford topological stacks

In this section we introduce a class of topological stacks which are counterparts of Deligne-Mumford stacks in algebraic geometry. We call them *Deligne-Mumford topological stacks*. In fact, there is a weaker notion, called *weak Deligne-Mumford topological stack*, which could also be regarded as topological counterpart of algebraic Deligne-Mumford stacks. The weak Deligne-Mumford topological stacks, however, could behave pathologically in certain situations, since they are a bit too general.

We begin with a definition which is modeled on the notion of a "slice" of a group action. Group actions for which slices exist will satisfy the following property.

Definition 14.1. Let G be a topological group acting continuously on a topological space X and let x be a fixed point of this action. We say this action is **mild** at x if every open neighborhood of x contains an invariant open neighborhood (that is, the invariant opens form a basis at x).

For example, any finite group action is mild at every fixed point. Any continuous action of a compact Lie group on a topological space also has this property, but in this paper we are only concerned with the discrete groups, so we will not get into that.

Definition 14.2. Let G be a discrete group acting on a topological space X. We say that this action is **properly discontinuous** if, for every $x \in X$, there is an I_x-invariant open neighborhood U of x such that:

- ◇ I_x acts mildly on U;
- ◇ for every $g \in G\backslash I_x$, $U \cap g(U) = \emptyset$.

Here I_x stands for the stabilizer group at x. Note that the mildness condition is automatic if I_x is finite.

[8] For example, the *suspension* construction usually fails for topological stacks.

Definition 14.3. A pretopological stack $\mathcal{X}$ is called a **weak Deligne-Mumford topological stack** if there is a chart $p\colon X \to \mathcal{X}$ that is a local homeomorphism. A weak Deligne-Mumford topological stack $\mathcal{X}$ is called a **Deligne-Mumford topological stack**, if for every point x in $\mathcal{X}$ and every open substack $\mathcal{U} \subseteq \mathcal{X}$ containing x there is an open substack $\mathcal{V} \subseteq \mathcal{U}$ containing x such that $\mathcal{V} \cong [V/I_x]$, for some topological space V with an action of I_x that is mild at the (unique) fixed point of V lying above x.

Weak Deligne-Mumford topological stacks are exactly topological stack for **LF**=local homeomorphisms. Weak Deligne-Mumford topological stacks form a full 2-subcategory of TopSt, which we denote by WeakDM. Deligne-Mumford topological stacks form a full 2-subcategory of WeakDM, which we denote by DM. The Yoneda functor identifies Top with a full 2-subcategory of DM.

Proposition 14.4. *Let $\mathcal{X}$ be a pretopological stack.*

- **i**. *If $\mathcal{X}$ is (weak) Deligne-Mumford, then so is every embedded substack of $\mathcal{X}$.*
- **ii**. *Let $\{\mathcal{U}_i\}_{i\in I}$ be a covering of $\mathcal{X}$ by open substacks. If every $\mathcal{U}_i$ is (weak) Deligne-Mumford, then so is $\mathcal{X}$.*
- **iii**. *The 2-category WeakDM is closed under 2-fiber products. (See Section 15 for the case of DM.)*

We can characterize étale groupoids whose quotient is Deligne-Mumford as follows.

Proposition 14.5. *Let $[R \rightrightarrows X]$ be an étale groupoid and set $\mathcal{X} = [X/R]$. The necessary and sufficient condition for $\mathcal{X}$ to be a Deligne-Mumford topological stack is that, for every $x \in X$ and every open U' containing x, there exist an open $x \in U \subseteq U'$ such that the $[R|_U \rightrightarrows U]$ is isomorphic to the action groupoid of an I_x-action on U.*

Proof. One implication is obvious. Assume now that $\mathcal{X}$ is Deligne-Mumford. By definition, there is a topological space V with a mild I_x action such that $[V/I_x]$ is equivalent to an open substack of $\mathcal{X}$ containing $p(x)$. This gives a local homeomorphism $q\colon V \to \mathcal{X}$. Since $p\colon X \to \mathcal{X}$ is a local homeomorphism, we may assume, after possibly shrinking V to a smaller I_x-invariant open neighborhood of x (that we can, since I_x acts mildly), that there is a lift $\tilde{q}\colon V \to X$ of q to X (as a pointed map). Since $q = p \circ \tilde{q}$ and p are local homeomorphism, $\tilde{q}$ is also a local homeomorphism. By shrinking V again around x, we may assume that $\tilde{q}$ is an open embedding. The image $U = \tilde{q}(V) \subseteq X$ is the open neighborhood of x we were after. We can make U as small as we want by shrinking V. $\square$

Corollary 14.6. *Let G be a discrete group acting on a topological space X. Then $[X/G]$ is Deligne-Mumford if an only if the action is properly discontinuous (Definition 14.2).*

Proof. If the action is properly discontinuous, $[X/G]$ is obviously Deligne-Mumford. Assume now that $[X/G]$ is Deligne-Mumford. Take an arbitrary point $x \in X$. Consider the action groupoid $[G \times X \rightrightarrows X]$. The open neighborhood U of x constructed in the Proposition 14.5 satisfies the property required in the definition 14.2. $\square$

Corollary 14.7. *A pretopological stack is Deligne-Mumford if and only if it can be covered by open substacks of the form $[X/G]$, where G is a discrete group acting properly discontinuously on a topological space X.*

Corollary 14.8. *Let G be a finite group acting on a topological space X. Then $[X/G]$ is a Deligne-Mumford topological stack.*

Proposition 14.9. *Let $[R \rightrightarrows X]$ be an étale groupoid such that the diagonal map $R \to X \times X$ is a closed map onto its image (where the image is endowed with the subspace topology from $X \times X$) and that, for every $x \in X$, the stabilizer group I_x is finite. Assume further that X is locally connected. Then $[X/R]$ is a Deligne-Mumford topological stack.*

Proof. Let x be an arbitrary point on X, and let $H = I_x$ be its stabilizer group (which is finite). First we consider a special case.

Special case. Assume our groupoid has the following special form: $R = \coprod_{h \in H} R_h$ such that R_1 is the identity section, and that the restriction of t and s to each R_h is an open embedding from R_h to X. Note that the image under s and t of every R_h contains x. Let V be an open neighborhood of x that is contained in every $s(R_h)$. Call the pre-image under s of V in R_h by V_h. Set $U' = \cap t(V_h)$. Then, U' is an invariant neighborhood of x, and so is the connected component U of x in U'. The restriction of R over U is a groupoid of the form $[H \times U \rightrightarrows U]$, such that the restriction of the source and target maps to each layer $\{h\} \times U$ of it are isomorphisms onto U. Lemma 7.7 implies that this groupoid is in fact the groupoid associated to an action of H on U.

General case. We reduce to the special case considered above. Recall that $H = I_x$ is a subset of R, and that the source map $s \colon R \to X$ is a local homeomorphism. Since H is finite, we can find a neighborhood W of x, and neighborhoods W_h for each $h \in H$, such that s induces a homeomorphism from W_h to W. Let $A = R - \cup W_h$. Since $\Delta \colon R \to X \times X$ is closed onto its image, $\Delta(A)$ is a closed subset of $Y := \Delta(R)$. Note that, by the construction of A, (x,x) is not in $\Delta(A)$, so $Y - \Delta(A)$ is an open neighborhood of x in Y. Hence, there is an open neighborhood U of x in X such that $U \times U$ is contained in $Y - \Delta(A)$, or equivalently, $U \times U$ does not intersect $\Delta(A)$. It is easy to check that the restriction R_U of R to U (which is defined to be $R_U = \Delta^{-1}(U \times U)$, with the same source and target maps) is a groupoid which satisfies the property required in the special case above. The reason for this is that, indeed, R_U can be identified with an open subset of $\coprod W_h$, and the source and target maps for R_U are the ones induced from $\coprod W_h \to X$. □

Proposition 14.9 can be rephrased as follows.

Proposition 14.10. *Let $\mathcal{X}$ be a locally connected (Definition 15.5) weak Deligne-Mumford topological stack. Assume every point in $\mathcal{X}$ has a finite inertia group, and that the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a closed map onto its image (see Definitions 4.4 and 4.5 and the ensuing example). Then $\mathcal{X}$ is a Deligne-Mumford stack.*

For an example of a weak Deligne-Mumford topological stack that is not Deligne-Mumford see Example 7.8. For more examples of Deligne-Mumford topological stacks see Section 19.

15. Fiber products of Deligne-Mumford topological stacks

Unfortunately, the 2-category of Deligne-Mumford topological stacks fails to be closed under fiber products. But the failure is not so dramatic. We will see that, under some mild extra hypotheses, fiber products of Deligne-Mumford topological stacks will again be Deligne-Mumford (Corollary 15.8). The counterexamples could safely be regarded as pathological.

Definition 15.1. We say that a pretopological stack is **locally discrete-quotient** if it can be covered by open substacks each of which equivalent to the quotient stack of a discrete group acting on a topological space.

A locally discrete-quotient stack is weak Deligne-Mumford, but it fails to be Deligne-Mumford in general (e.g. pick a non-properly discontinuous group action and take its quotient).

Definition 15.2. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map of locally discrete-quotient stacks. We say f is **controlable**, if for every point $y \in \mathcal{Y}$, there is an open substack $\mathcal{V}$ around y and an open substack $\mathcal{U}$ around $f(y)$ such that f maps $\mathcal{V}$ into $\mathcal{U}$ and

- there are presentation $\mathcal{V} = [V/H]$ and $\mathcal{U} = [U/G]$ as quotient stacks by discrete group actions;
- for a suitable choice of such presentations, the restriction $f|_{\mathcal{V}}\colon \mathcal{V} \to \mathcal{U}$ is induced by a group homomorphism $\varphi\colon H \to G$ and a φ-equivariant map $V \to U$.

Remark that, it is not true in general that a map of quotient stacks $[V/H] \to [U/G]$ comes from a map of action groupoids $[H \times V \rightrightarrows V] \to [G \times U \rightrightarrows U]$. Also it is *not* true in general that a map of action groupoids $[H \times V \rightrightarrows V] \to [G \times U \rightrightarrows U]$ is induced by a group homomorphism $\varphi\colon H \to G$ and a φ-equivariant map $V \to U$.

If $f\colon \mathcal{Y} \to \mathcal{X}$ is controlable and $\mathcal{Y}$ is a Deligne-Mumford topological stack, then it can be shown that the second condition of the above definition holds for the presentation $\mathcal{V} = [V/I_x]$ too.

Proposition 15.3. *Consider a diagram*

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"'] & \mathcal{X} \end{tikzcd}$$

of pretopological stacks, and assume $\mathcal{Y}$ and $\mathcal{Z}$ are locally discrete-quotient stacks. Then so is $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$. In particular, the 2-category of locally discrete-quotient stacks is closed under 2-fiber products.

Proof. We may assume that $\mathcal{Y} = [Y/H]$ and $\mathcal{Z} = [Z/K]$ are quotient stacks, where H and K are discrete groups, and that the above diagram is induced from a diagram of groupoids

$$\begin{tikzcd} & {[H \times Y \rightrightarrows Y]} \arrow[d, "F"] \\ {[K \times Z \rightrightarrows Z]} \arrow[r, "G"'] & {[R \rightrightarrows X]} \end{tikzcd}$$

Before proceeding with the proof, we fix some notations. Let $F_0\colon Y \to X$ and $F_1\colon H \times Y \to R$ denote the structure maps of F. For each $h \in H$, let $F_h\colon Y \to R$ be the composition

$$Y \xrightarrow{\sim} \{h\} \times Y \hookrightarrow H \times Y \xrightarrow{F_1} R.$$

Define $G_0\colon Z \to X$, $G_1\colon K \times Z \to R$, and $G_k\colon Z \to R$ similarly.

The construction of Section 9 shows that the the fiber product $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$ is the quotient stack of the groupoid $[P_1 \rightrightarrows P_0]$, where $P_0 = Y \times_{\mathcal{X}} Z = (Y \times Z) \times_{X \times X} R$ and $P_1 = (H \times K) \times P_0$. The source map is simply the projection onto the second factor. This already suggests that $[P_1 \rightrightarrows P_0]$ is the action groupoid of an action of $H \times K$ on P_0. We give the action explicitly, using the description of the target map of the groupoid $[P_1 \rightrightarrows P_0]$ given in Section 9. Let $(h,k) \in H \times K$ be an arbitrary element. The action of (h,k) on $P_0 = (Y \times Z) \times_{X \times X} R$ sends a triple (y,z,r) to

$$\big(\,h(y),\, k(z),\, [F_{h^{-1}}\big(h(y)\big)] \cdot [r] \cdot [G_k(z)]\,\big)$$

where the elements in the square brackets belong to R, and $\cdot$ stands for composition of composable arrows in R. This is easily seen to be a group action. $\square$

The above proposition implies that the fiber product of Deligne-Mumford topological stacks is always a locally discrete-quotient stack. The next proposition tells us when this locally discrete-quotient stack is Deligne-Mumford.

Theorem 15.4. *Consider a diagram*

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"'] & \mathcal{X} \end{tikzcd}$$

of locally discrete-quotient stacks, and assume $\mathcal{Y}$ and $\mathcal{Z}$ are Deligne-Mumford. Also, assume that f and g are controlable. Then the 2-fiber product $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$ is a Deligne-Mumford topological stack.

Proof. We may assume that $\mathcal{Y} = [Y/H]$ (resp. $\mathcal{Z} = [Z/K]$), for some topological space Y (resp. Z) with a properly discontinuous H (resp. K) action (Corollary 14.6), and that $\mathcal{X} = [X/D]$, where D is a discrete group acting on X. We may also assume that our diagram of stacks is induced from the following diagram of groupoids

$$\begin{tikzcd} & {[H \times Y \rightrightarrows Y]} \arrow[d, "F=(\varphi,F_0)"] \\ {[K \times Z \rightrightarrows Z]} \arrow[r, "G=(\psi,G_0)"'] & {[D \times X \rightrightarrows X]} \end{tikzcd}$$

where the vertical map is induced from a group homomorphism $\varphi\colon H \to D$, and a φ-equivariant map $F_0\colon Y \to X$, and the horizontal map is induced from a group homomorphism $\psi\colon K \to D$, and a ψ-equivariant map $G_0\colon Z \to X$. Set $P := Y \times_{\mathcal{X}} Z = (Y \times Z) \times_{X \times X} (D \times X)$ (this is what we called P_0 is Section 9). In the course of proof of Proposition 15.3 we showed that there is an action of $H \times K$ on P such that the quotient stack $[P/H \times K]$ is equivalent to $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$. It is enough to show that this action is properly discontinuous. Pick an arbitrary point $p_0 \in P = (Y \times$

$Z) \times_{X \times X} (D \times X)$. This point can be represented as $p_0 = (y_0, z_0, x_{0s}, x_{0t}, d_0, x_0)$, where $y_0 \in Y$, $z_0 \in Z$, $x_{0s}, x_{0t}, x \in X$, and $d_0 \in D$. Note that, $F_0(y_0) = x_{0s}$, $G_0(z_0) = x_{0t}$, $x_0 = x_{0s}$, and $d_0(x_{0s}) = x_{0t}$, where $d_0 \colon X \to X$ is the automorphism coming from the action of D on X. Denote by I_{y_0}, I_{z_0}, $I_{x_{0s}}$ and $I_{x_{0t}}$ the stabilizer groups of the corresponding group actions at the respective points. Observe that, since $d(x_{0s}) = x_{0s}$, conjugation by d_0 maps $I_{x_{0s}}$ isomorphically to I_{x_0t}. So we *identify* $I_{x_{0s}}$ and $I_{x_{0t}}$ via this isomorphism and call them both I.

Having made these remarks, we can now shorten the notation for a point $p = (y, z, , x_s, x_t, d, x) \in P$ by simply denoting it by $p = (y, z, d)$.

We have natural group homomorphisms $I_{y_0} \to I$ and $I_{z_0} \to I$, induced by φ and ψ, respectively. We claim the following:

- **a**. The stabilizer group at p_0 of the action of $H \times K$ on P is equal to the fiber product $I_{y_0} \times_I I_{z_0} \subseteq H \times K$. (Note that
$$I_{y_0} \times_I I_{z_0} = \{(h, k) \in I_{y_0} \times I_{z_0} | \;\; d_0 \; \psi(k) \; d_0^{-1} = \varphi(h)\}.)$$
- **b**. There exists an $I_{y_0} \times_I I_{z_0}$-invariant open neighborhood U of p_0 in P such that the action of $I_{y_0} \times_I I_{z_0}$ on P is mild at p_0, and for any $(h, k) \in H \times K$ not in $I_{y_0} \times_I I_{z_0}$, $(h, k)(U) \cap U = \emptyset$.

Such open neighborhood U of p_0 satisfies the property required in Definition 14.2, hence, the proposition follows.

Proof of (**a**). If we decipher the description of the action of $H \times K$ given in Proposition 15.3 in the case where the groupoid $[R \rightrightarrows X]$ is the action groupoid $[D \times X \rightrightarrows X]$, we see that an arbitrary $(h, k) \in H \times K$ sends $p = (y, z, d)$ to
$$\big(h(y), \, k(z), \, \varphi(h^{-1}) \; d \; \psi(k) \big)$$
This immediately implies (**i**).

Proof of (**b**). Set
$$U = \{(y, z, d) \in P | \; d = d_0\}.$$
This an open neighborhood of $p_0 = (y_0, z_0, d_0)$. By definition of the fiber product, $I_{y_0} \times_I I_{z_0}$ is the set of pairs $(h, k) \in H \times K$ such that $\varphi(h^{-1}) d_0 \psi(k) = d_0$. So if (h, k) *is not* in $I_{y_0} \times_I I_{z_0}$, the third component of any element in $(h, k)(U)$ is different from d_0, so $(h, k)(U) \cap U = \emptyset$. Similarly, if (h, k) *is* in $I_{y_0} \times_I I_{z_0}$, we have $(h, k)(U) = U$. All that remains to be proved is that the action of $I_{y_0} \times_I I_{z_0}$ on U is mild. To prove this, note that we have an $I_{z_0} \times_I I_{y_0}$-equivariant isomorphism $U \xrightarrow{\sim} Y \times Z$ (forgetting the factor d). Since the actions of I_{y_0} on Y and I_{z_0} on Z are mild, so is the action of $I_{z_0} \times I_{y_0}$ on $Y \times Z$. In particular, the action of the subgroup $I_{z_0} \times_I I_{y_0}$ is also mild. Therefore, the action of $I_{z_0} \times_I I_{y_0}$ on U is mild. The proof of the proposition is now complete. ☐

The above proposition is useless unless we know how to determine if a map of locally discrete-quotient stacks is controlable. The Proposition 15.7 tell us that it is the case in quite general situations.

Definition 15.5. Let $\mathcal{X}$ be a pretopological stack. We say $\mathcal{X}$ is **locally connected**, if there exists a chart $p \colon X \to \mathcal{X}$ such that X is a locally connected topological space (also see Section 18.1).

Lemma 15.6. *Let $\mathcal{X}$ be a locally connected weak Deligne-Mumford stack. Then for any étale chart $p \colon X \to \mathcal{X}$, X is locally connected.*

Proof. See Lemma 18.9. □

Proposition 15.7. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map of stacks, where $\mathcal{X}$ is a locally discrete-quotient stack and $\mathcal{Y}$ is Deligne-Mumford. Then f is controlable, if at least one of the following holds:*

- *$\mathcal{X}$ is a topological space;*
- *$\mathcal{Y}$ is locally connected (see Definition 15.5);*
- *$\mathcal{Y}$ has a finitely generated inertia group at every point.*

Proof. The case where $\mathcal{X}$ is a topological space is obvious. We do the other two cases.

Pick a point y on $\mathcal{Y}$, and let $H = I_y$. We may assume $\mathcal{Y} = [Y/H]$, where Y has an H-action with a unique fixed point y_0, lying above y, at which H acts mildly. Assume that either Y is connected and locally connected (Lemma 15.6), or that H is finitely generated. We may also assume that $\mathcal{X} = [X/G]$, with G discrete. Finally, by shrinking Y around y_0, we may assume that the composite map $Y \to \mathcal{X}$ lifts to X; that is, f is induced by a map of action groupoids $F\colon [H \times Y \rightrightarrows Y] \to [G \times X \rightrightarrows X]$. The notation $F_0\colon Y \to X$ and $F_1\colon H \times Y \to G \times X$ should be clear. We consider the two cases separately.

Y is connected. Since Y is connected, under the map $F_1\colon H \times Y \to G \times X$, every layer $\{h\} \times Y$, $h \in H$, should factor through a layer $\{g\} \times X$, for some $g \in G$. This gives a map $\varphi\colon H \to G$ that induces the original map $F\colon [H \times Y \rightrightarrows Y] \to [G \times X \rightrightarrows X]$ of groupoids. It is easily seen that φ is a group homomorphism, and the map $Y \to X$ is φ-equivariant.

H is finitely generated. Let $x_0 = F_0(y_0)$ be the image of y_0 in X. Let $h_1, h_2, \cdots, h_n$ be generators for H. For each $i = 1, 2, \cdots, n$, the map $F_1\colon H \times Y \to G \times X$ maps the point (h_i, y_0) to some (g_i, x_0), $g_i \in G$. Let $V_i \subseteq Y$ be a neighborhood of y_0 such that $F_1(\{h_i\} \times V_i) \subseteq \{g_i\} \times X$. Let V be an H-invariant neighborhood of y_0 contained in $\bigcap U_i$. Then $F_1(\{h_i\} \times V) \subseteq \{g_i\} \times X$, for every i. But since F_1 comes from a map of groupoids and h_i generate H, it follows that, for every $h \in H$, we have $F_1(\{h\} \times V_i) \subseteq \{g\} \times X$, for some $g \in G$. That is, we have a map $\varphi\colon H \to G$ that induces the original map $F\colon [H \times Y \rightrightarrows Y] \to [G \times X \rightrightarrows X]$ of groupoids. It is easily seen that φ is a group homomorphism, and the map $Y \to X$ is φ-equivariant. □

Corollary 15.8. *Consider a diagram*

$$\begin{tikzcd} & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "g"'] & \mathcal{X} \end{tikzcd}$$

of locally discrete-quotient stacks, and assume $\mathcal{Y}$ and $\mathcal{Z}$ are Deligne-Mumford. Assume at least one of the following holds:

- *$\mathcal{X}$ is a topological space;*
- *each of $\mathcal{Y}$ and $\mathcal{Z}$ is either locally connected (see Definition 15.5) or has finitely generated inertia groups.*

Then $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$ is a Deligne-Mumford topological stack.

One can construct examples where a 2-fiber product of Deligne-Mumford topological stacks is not Deligne-Mumford. I will give a flavor of how such an example is constructed, but refrain from giving the messy details. Let $\mathbb{F}$ be the free group on countably many generators. Take $\mathcal{X} = \mathcal{B}\mathbb{F} = [*/\mathbb{F}]$ and $\mathcal{Z} = *$. The interesting one is $\mathcal{Y}$, whose construction is a bit more involved. Let $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Set $Y = \mathbb{F} \times A/\mathbb{F} \times \{0\}$ with the topology that is discrete on the complement of the distinguished point (which we call O), and has a local basis at O consisting of "balls of radius $\frac{1}{n}$" (these are all point, in various copies of A appearing in Y, whose distance to 0 is less than $\frac{1}{n}$). Notice that this topology is weaker than the quotient topology on $\mathbb{F} \times A/\mathbb{F} \times \{0\}$. There is an obvious *mild* "rotation" action of $\mathbb{F}$ on Y fixing O and acting freely on the rest. Define $\mathcal{Y} = [Y/\mathbb{F}]$. Outside the distinguished point, $\mathcal{Y}$ is equivalent to the discrete set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$. At the distinguished point $\mathcal{Y}$ looks like $\mathcal{B}\mathbb{F}$ (i.e. $\Gamma_O \cong \mathcal{B}\mathbb{F}$). What is interesting is the way the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is "approaching" $\mathcal{B}\mathbb{F}$.

The fun part is to construct an *uncontrolable* map $f\colon \mathcal{Y} \to \mathcal{X}$ which satisfies certain nice properties. I will not get into that, but the outcome is that, the fiber product $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$ (which can be thought of as the fiber of f over a point) will turn out to be a quasitopological space that is not a topological space. In particular, it can not be a Deligne-Mumford topological stack.

This example also shows that a covering spaces of a non locally connected Deligne-Mumford topological stack is not necessarily a Deligne-Mumford stack (see Proposition 18.25), because in this case $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$ is a covering space of $\mathcal{Z}$.

16. Gluing topological stack along substacks

Throughout this section we fix a class **LF** of local fibrations (Section 13.1). By a substack we mean an embedded substack.

In point set topology, given an equivalence relation on a topological space, one can always construct a quotient space, satisfying the expected universal property. It turns out that in the world of pretopological stacks this is far from being true; even sticking to the special case of Deligne-Mumford does not help much. More depressing is the fact that even the good old quotients of topological *spaces* may loose their universal property when viewed in the 2-category of topological stack. So we have got to be very careful with our intuition. Fortunately, by choosing the right class of local fibrations, we can fine-tune our category of topological stacks so that our desired gluing and quotient constructions can be performed within the realm of topological stacks.

It turns out that, for the purpose of homotopy theory, the required conditions on **LF** are quite mild (Section 17).

The following lemma is essentially due to Angelo Vistoli.

Proposition 16.1. *Let L, Y and Z be topological spaces. Suppose we are given a push-out diagram in the category of topological spaces*

$$\begin{tikzcd} L \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & Y \arrow[d] \\ Z \arrow[r] & Y \vee_L Z \end{tikzcd}$$

with i and j embeddings. Suppose further that $i\colon L \hookrightarrow Y$ is LTC (see Definition 13.3). Then, this diagram remains a push-out in the 2-category of topological stacks. In other words, for every topological stack $\mathcal{X}$, given a pair of maps $f\colon Y \to \mathcal{X}$ and $g\colon Z \to \mathcal{X}$ that are compatible along L (i.e. we are given an identification $\varphi\colon f|_L \Rightarrow g|_L$), f and g can be glued in a unique (up to a unique identification) way to a map $f \vee_L g\colon Y \vee_L Z \to \mathcal{X}$.

Before giving the proof, let us make the statement precise. Let $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{L}$ be pretopological stacks, and $i\colon \mathcal{L} \hookrightarrow \mathcal{Y}$ and $j\colon \mathcal{L} \hookrightarrow \mathcal{Z}$ be embeddings. Define the groupoid $\mathrm{Glue}(\mathcal{Y},\mathcal{Z},\mathcal{L};\mathcal{X})$ of *gluing data* as follows:

$$\mathrm{Ob}\,\mathrm{Glue}(\mathcal{Y},\mathcal{Z},\mathcal{L};\mathcal{X}) = \left\{ \ (f,g,\varphi) \quad \middle| \quad f\colon \mathcal{Y}\to\mathcal{X},\ g\colon \mathcal{Z}\to\mathcal{X},\ f|_{\mathcal{L}} \overset{\varphi}{\Rightarrow} g|_{\mathcal{L}} \ \right\}$$

$$\mathrm{Mor}_{\mathrm{Glue}(\mathcal{Y},\mathcal{Z},\mathcal{L};\mathcal{X})}\big((f,g,\varphi),(f',g',\varphi')\big) = \left\{ \begin{array}{lll} (\alpha,\beta) & \Big| & f \overset{\alpha}{\Rightarrow} f',\ g \overset{\beta}{\Rightarrow} g' \text{ identifications} \\ & & \text{such that} \quad \begin{tikzcd} f|_L \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, "\alpha|_{\mathcal{L}}"'] \arrow[dr, phantom, "\circlearrowleft"] & g|_L \arrow[d, Rightarrow, "\beta|_{\mathcal{L}}"] \\ f'|_{\mathcal{L}} \arrow[r, Rightarrow, "\varphi'"'] & g'|_{\mathcal{L}} \end{tikzcd} \end{array} \right\}$$

Proposition 16.1 follows from the following theorem.

Theorem 16.2. *Notation being as in Proposition 16.1, consider the natural restriction functor*

$$\varrho\colon \mathrm{Hom}_{\mathsf{St}}(Y \vee_L Z, \mathcal{X}) \to \mathrm{Glue}(Y,Z,L;\mathcal{X})$$
$$H \mapsto (H|_{\mathcal{Y}}, H|_{\mathcal{Z}}, \mathrm{id}).$$

This functor is an equivalence of groupoids. Furthermore, if we do not assume that $i\colon L \to Y$ is LTC, then ϱ is still fully faithful.

Proof. Fix an **LF** chart $p\colon X \to \mathcal{X}$. Let $[R \rightrightarrows X]$ be the corresponding groupoid.

Proof of faithfulness. Let $H, H'\colon Y \vee_L Z \to \mathcal{X}$ be morphisms, and $\Psi_1, \Psi_2\colon H \Rightarrow H'$ identifications between them. Assume $\Psi_1|_Y = \Psi_2|_Y$ and $\Psi_1|_Z = \Psi_2|_Z$. We have to show that $\Psi_1 = \Psi_2$. Since $\mathcal{X}$ is a stack, it is enough to do this locally. So, after replacing $Y \vee_L Z$ by an open covering, we may assume that H and H' lift to X. Denote the lifts by $h, h'\colon Y \vee_L Z \to X$. (Recall that, a *lift* of H is a pair (h,γ), with $h\colon Y \vee_L Z \to X$ and $\gamma\colon p\circ h \Rightarrow H$; we supress γ in the notation.) This gives a map $(h,h')\colon Y \vee_L Z \to X \times X$. The 2-isomorphisms Ψ_1 and Ψ_2 precisely correspond to lifts of this map to R:

$$\begin{tikzcd} & R \arrow[d] \\ Y \vee_L Z \arrow[ur, bend left, "\psi_1"] \arrow[ur, bend right, "\psi_2"'] \arrow[r, "{(h,h')}"'] & X \times X \end{tikzcd}$$

Since $\Psi_1|_Y = \Psi_2|_Y$, we have $\psi_1|_Y = \psi_2|_Y$. Similarly, we have $\psi_1|_Z = \psi_2|_Z$. Therefore, since $Y \vee_L Z$ is a push out in the category of topological spaces, we have $\psi_1 = \psi_2$. This implies that $\Psi_1 = \Psi_2$.

Proof of fullness. Let $H, H'\colon Y \vee_L Z \to \mathcal{X}$ be morphisms. An arrow from $\varrho(H)$ to $\varrho(H')$ in the groupoid $\mathrm{Glue}(Y,Z,L;\mathcal{X})$ is a pair of identifications $\Psi_Y\colon H|_Y \Rightarrow H'|_Y$

and $\Psi_Z \colon H|_Z \Rightarrow H'|_Z$ such that $\Psi_Y|_L = \Psi_Z|_L$. We have to show that there exists an identification $\Psi \colon H \Rightarrow H'$ such that $\Psi|_Y = \Psi_Y$ and $\Psi|_Z = \Psi_Z$.

Having proved the uniqueness of Ψ in the previous part, it is now enough to prove the existence locally (because $\mathcal{X}$ is a stack). So, again, we may assume that we have lifts $h, h' \colon Y \vee_L Z \to X$ of H and H'. We want to construct a lift ψ as in the following diagram:

$$\begin{tikzcd} & R \arrow[d] \\ Y \vee_L Z \arrow[ur, dotted, "\psi"] \arrow[r, "(h,h')"'] & X \times X \end{tikzcd}$$

We have a lift of $\psi_Y : Y \to R$ of $(h,h')|_Y$ obtained from the 2-isomorphism Ψ_Y, and a similar one $\psi_Z : Z \to R$ obtained from Ψ_Z. Since the restrictions of Ψ_Y and Ψ_Z to L are equal, we have $\psi_Y|_L = \psi_Z|_L$. Therefore, ψ_Y and ψ_Z glue to a map $\psi \colon Y \vee_L Z \to R$. The 2-isomorphism $\Psi \colon H \Rightarrow H'$ corresponding to ψ is what we were looking for.

This much of the proof works for any pretopological stack $\mathcal{X}$, without LTC assumption on $i \colon L \hookrightarrow Y$.

Proof of essential surjectivity. Assume we are given a gluing datum

$$(f, g, \varphi) \in \mathrm{Ob}\,\mathrm{Glue}(Y, Z, L; \mathcal{X}).$$

We have to show that there exists $H \colon Y \vee_L Z \to \mathcal{X}$ such that $(H|_Y, H|_Z, \mathrm{id})$ is isomorphic to (f, g, φ) in $\mathrm{Glue}(Y, Z, L; \mathcal{X})$.

In view of the previous parts, it is enough to prove the existence locally. In other words, we have to show that for every point $t \in L$, there is an open neighborhood U of t in $Y \vee_L Z$ over which the gluing is defined. (If $t \in (Y \vee_L Z) \backslash L$ the existence is obvious.) By shrinking Y and Z around t we may assume that that f and g have lifts $\tilde{f} \colon Y \to X$ and $\tilde{g} \colon Z \to X$. The identification $f|_L \overset{\varphi}{\Rightarrow} g|_L$ corresponds to a lift ϕ as in the following diagram:

$$\begin{tikzcd} & R \arrow[d] \\ L \arrow[ur, "\phi"] \arrow[r, "(\tilde{f}|_L, \tilde{g}|_L)"'] & X \times X \end{tikzcd}$$

Consider the following commutative diagram:

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d, "i"'] & R \arrow[d, "s"] \\ Y \arrow[ur, dotted, "F"] \arrow[r, "\tilde{f}"'] & X \end{tikzcd}$$

The map $i \colon L \hookrightarrow Y$ is LTC and s is LF, so by shrinking Y around t we can fill the dotted line. Shrink Z accordingly, so that Y and Z are intersect L in the same open set. (This can be done, since L is a subspace of Z.) The maps $t \circ F \colon Y \to X$ and

$\tilde{g}\colon Z \to X$ now agree (on the nose) along L, so they glue to a map $h\colon Y \vee_L Z \to X$. The composition $H = p \circ h$ is the desired gluing of f and g. $\square$

It is an interesting question what kinds of colimits in Top remain a "colimit" in TopSt. It is easy to construct examples where this fails.

Example 16.3. Consider the push-out diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "x\mapsto -x"] \arrow[d, "\mathrm{id}"'] & \mathbb{R}^2 \arrow[d] \\ \mathbb{R}^2 \arrow[r] & \mathbb{R}^2/\mathbb{Z}_2 \end{tikzcd}$$

This diagram is no longer a push-out diagram in TopSt. For example, take $\mathcal{X} = [\mathbb{R}^2/\mathbb{Z}_2]$ and let $f = g\colon \mathbb{R}^2 \to \mathcal{X}$ be the obvious map. Then, there is no induced map on the colimit $\mathbb{R}^2/\mathbb{Z}_2 \to \mathcal{X}$. (Hint: if such map existed, it would have a lift to $\mathbb{R}^2$ around a small enough neighborhood of $0 \in \mathbb{R}^2/\mathbb{Z}_2 = \mathbb{R}^2$.)

The next thing we want to consider is the question of existence of push outs in the 2-category of topological stacks. We formulate the question in the following form.

Gluing Problem. Let $\mathcal{L}$, $\mathcal{Y}$ and $\mathcal{Z}$ be topological stacks. Suppose we are given a diagram

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & \mathcal{Y} \\ \mathcal{Z} & \end{tikzcd}$$

with i and j embeddings. Can we can glue $\mathcal{Y}$ and $\mathcal{Z}$ along $\mathcal{L}$ to obtain a topological stack $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$?

A few remarks on this gluing problem are in order. First of all, the assumptions on the maps i and j being embeddings is very crucial. Not all push-outs exist in TopSt. For instance, if $\mathcal{X}$ is a non-trivial $\mathbb{Z}_2$-gerbe over S^1, it can be shown that $I \times \mathcal{X}/\{1\} \times \mathcal{X}$ can not exist in any reasonable sense. In particular, *suspensions* of stacks do not always exist.

The second thing that should be made clear is what it means to glue pretopological stacks! We require the glued pretopological stack $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ to satisfy the following properties:

G1 There are substacks $\mathcal{Y}'$ and $\mathcal{Z}'$ and $\mathcal{L}' := \mathcal{Y}' \cap \mathcal{Z}'$ of $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$, together with equivalences $\mathcal{Y} \xrightarrow{\sim} \mathcal{Y}'$, $\mathcal{Z}' \xrightarrow{\sim} \mathcal{Z}$ and $\mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ that fit into a commutative 2-cell

$$\begin{tikzcd} & \mathcal{Y} \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{L} \arrow[ur, "i"] \arrow[rr] \arrow[dr, "j"'] & \arrow[d, phantom, "\Downarrow"] & (\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}) \\ & \mathcal{Z} \arrow[ur] & \end{tikzcd}$$

G2 The equivalences in **G1** induce an equivalence

$$(\mathcal{Y}\backslash\mathcal{L})\coprod(\mathcal{Z}\backslash\mathcal{L}) \xrightarrow{\sim} (\mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z})\backslash\mathcal{L}'.$$

When $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{L}$ are topological spaces, it is not difficult (but not so trivial either) to show that these conditions characterize $\mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z}$ up to homeomorphism. In the case of pretopological stacks, however, it is presumably not true in general. So, in this generality, there might be several ways of gluing $\mathcal{Y}$ and $\mathcal{Z}$ along $\mathcal{L}$. We will see that, once we impose certain conditions on the stacks and on the embeddings i and j, then we can state some uniqueness results.

The reason for formulating the gluing problem using the rather strange looking conditions **G1** and **G2**, instead of a universal property, is that, first of all, these two conditions are obviously necessary, and second of all, a gluing $\mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z}$ always exists in this sense (but it may not satisfy the universal property).

The question whether $\mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z}$ does indeed satisfy the universal property of a push out can then be discussed as a separate problem (see Theorem 16.9).

Theorem 16.4. *Let $\mathcal{L}$, $\mathcal{Y}$ and $\mathcal{Z}$ be pretopological stacks. Suppose we are given a diagram*

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & \mathcal{Y} \\ \mathcal{Z} & \end{tikzcd}$$

where i and j are embeddings. Then, $\mathcal{Y}$ and $\mathcal{Z}$ can be glued along $\mathcal{L}$ to give a pretopological stack $\mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z}$ (see axioms **G1** *and* **G2** *above). Furthermore, if $\mathcal{Y}$ and $\mathcal{Z}$ are topological and admit neat LF charts with respect to $\mathcal{L}$ (see Definition 16.5 below), then $\mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z}$ admits a chart $X \to \mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z}$ whose restrictions over both $\mathcal{Y}$ and $\mathcal{Z}$ are LF. (See Remark 16.8 for a discussion about when $\mathcal{Y}\vee_{\mathcal{L}}\mathcal{Z}$ becomes topological.)*

To prove this theorem we need some preparation.

Definition 16.5. Recall that we have fixed a class **LF** of local fibrations. Let Y be a topological space and $L \subseteq Y$ a subspace. An epimorphic local fibration $f\colon L' \to L$ is said to be **extendable**, if there is a topological space Y' containing L' as a subspace, and an epimorphic local fibration $\tilde{f}\colon Y' \to Y$ extending f, such that $\tilde{f}^{-1}(L) = L'$. If such an extension is only possible after replacing L' by an open covering, we say f is **locally extendable**. We say (Y, L) is a **neat** pair (or L is a neat subspace of Y), if every local fibration $f\colon L' \to L$ is locally extendable.

For a pair $(\mathcal{Y}, \mathcal{L})$ of topological stacks, we say an LF chart $Y \to \mathcal{Y}$ is neat, if the corresponding subspace pair (Y, L) is neat.

Being neat is not a very strong condition, as we see in the following example.

Example 16.6.

1. The notions of locally extendable and neat are both local on the target, in the sense that, given a family of open subspaces $\{U_i\}$ of Y whose union contains L, we can check local extendability, or neatness, by doing so for every $L \cap U_i \hookrightarrow U_i$.
2. Every subspace $L \subseteq Y$ that is a local retract is neat.

To see this, we use part (**1**) to reduce to the case where $L \subseteq Y$ is a retract. Let $r\colon Y \to L$ be the retraction map. Then, for any local fibration $f\colon L' \to L$, the projection map $Y \times_L L' \to Y$ is a local fibration that extends f. Note that $L' \hookrightarrow Y \times_L L'$ is again a retract.

3. Assume **LF** is (contained in) the class of locally cartesian maps (Example 13.1.**3**) or local homeomorphisms (Example 13.1.**6**), then any subspace $L \subseteq Y$ is neat.

4. Let **LF** be the class of all epimorphisms. Then every subspace $L \subseteq Y$ is neat.

To see this, let $f\colon L' \to L$ be an epimorphism. Since local extendability is local on the target by part (**1**), we may assume f admits a section $s\colon L \to L'$. We can now take $Y' = L' \vee_L Y$, and extend f over to Y' in the obvious way.

Exercise. Show that if $\mathcal{Y}$ is a weak Deligne-Mumford topological stack and $\mathcal{L} \hookrightarrow \mathcal{Y}$ a neat substack, then the condition of Definition 16.5 is satisfied for *every* étale chart $Y \to \mathcal{Y}$. (Hint: adopt the proof of Lemma 18.9.)

The following simple lemma is used in the proof of Theorem 16.4.

Lemma 16.7.

i. *Let Y be a topological space and $L \subseteq Y$ a subspace. Let $\{U_i\}$ be an open covering of L. Then $\coprod U_i \to L$ is extendable. If (Y, L) is a neat pair, then the extension can also be chosen to be neat.*

ii. *Let $\mathcal{L} \hookrightarrow \mathcal{Y}$ be an embedding of topological stacks, and let $Y \to \mathcal{Y}$ be a neat chart with $L \subseteq Y$ the (neat) subspace corresponding to $\mathcal{L}$. Let $L' \to L$ be an epimorphic local fibration, and think of it as a chart for $\mathcal{L}$. Then $L' \to \mathcal{Y}$ is locally extendable to an LF chart $Y' \to \mathcal{Y}$ for $\mathcal{Y}$.*

Proof of part (**i**). For each U_i in the covering find an open V_i in Y such that $V_i \cap L = U_i$, and set $Y' = (\coprod V_i) \coprod (Y \backslash L)$.

Proof of part (**ii**). Since $L \subseteq Y$ is neat, we can assume, after possibly replacing L' by an open covering, that $L' \to Y$ extends to an LF epimorphism $Y' \to Y$, for some topological space Y' containing L' as a subspace. Composing this map with the chart $Y \to \mathcal{Y}$, give the desired extension of $L' \to \mathcal{L}$. □

Proof of Theorem 16.4. We assume that $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{L}$ are topological stacks, and that $\mathcal{Y}$ and $\mathcal{Z}$ admit nice charts with respect to $\mathcal{L}$. The pretopological case follows by taking **LF** to be the class of all epimorphisms (see also Example 16.6.**4**).

Pick LF charts $p\colon Y \to \mathcal{Y}$ and $q\colon Z \to \mathcal{Z}$ such that the (invariant) subspaces $L_1 \subseteq Y$ and $L_2 \subseteq Z$ corresponding to $\mathcal{L}$ are neat. We would like to glue Y to Z along these subspaces to build a chart for the would-be $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$. But the problem is that L_1 may not be homeomorphic to L_2. Thanks to Lemma 16.7, we can overcome this problem as follows.

For any two LF charts L_1 and L_2 for $\mathcal{L}$, there exists a third chart L which factors through L_1 and L_2 via local fibrations (e.g. take $L = L_1 \times_{\mathcal{L}} L_2$). By Lemma 16.7.**ii**, we may replace L by some open cover and assume that it extends to a chart for $\mathcal{Y}$. By refining the open cover, we may assume that L extends to a chart for $\mathcal{Z}$ as well (here we used Lemma 16.7.**i**).

From now on, we will assume that L_1 and L_2 are homeomorphic. Let $[S \rightrightarrows Y]$ and $[T \rightrightarrows Z]$ be the groupoid associated to the chart $Y \to \mathcal{Y}$and $Z \to \mathcal{Z}$. Denote the restriction of S to L_1 by $[R_1 \rightrightarrows L_1]$ and the restriction of T to L_2 by $[R_2 \rightrightarrows L_2]$. We have $R_1 \cong L_1 \times_{\mathcal{L}} L_1$ and $R_2 \cong L_2 \times_{\mathcal{L}} L_2$, so the homeomorphism between L_1 and L_2 induces an isomorphism between the groupoids $[R_1 \rightrightarrows L_1] \xrightarrow{\sim} [R_2 \rightrightarrows L_2]$. We can now glue the groupoids $[S \rightrightarrows Y]$ and $[T \rightrightarrows Z]$ along this isomorphism. The quotient stack of this new groupoid is the desired gluing $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$. □

There are two issues with the gluing of topological stacks: 1) It is not guaranteed that when we glue topological stacks along neat substacks we obtain topological stacks; 2) The pretopological stack $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ may not satisfy the universal property of push-out.

We discuss the first issue in the following remark.

Remark 16.8. The source and target maps of the groupoid constructed in the proof of the theorem are not expected to be LF in general. However, they are very close to be so. For instance, the source map is a map of the form $s\colon X \to Y \vee_L Z$ where both restrictions $s|_Y\colon s^{-1}(Y) \to Y$ and $s|_Z\colon s^{-1}(Z) \to Z$ are LF. Although this in general does not imply that s is LF, but it is quite likely that in specific examples one can prove by hand that this is the case. Especially, note that usually L is a local retract in both Y and Z.

If we take *LF* to be the class of local homeomorphisms, or locally cartesian maps (Example 13.1.**3**), then it *is* true that s and t are again LF. So in this case the glued stack is again topological. Note also that in these cases every subspace is automatically neat (Example 16.6).

Now, we turn to the universal property of $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$. It follows from the definition of $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ (see axioms **G1** and **G2**) that there is a natural restriction functor

$$\varrho\colon \mathrm{Hom}_{\mathsf{St}}(\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}, \mathcal{X}) \to \mathrm{Glue}(\mathcal{Y}, \mathcal{Z}, \mathcal{L}; \mathcal{X}),$$

where $\mathrm{Glue}(\mathcal{Y}, \mathcal{Z}, \mathcal{L}; \mathcal{X})$ is the groupoid of gluing data defined in page 16. We say $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ satisfies the *universal property* if this functor is an equivalence of groupoids.

We generalize Theorem 16.2 as follows.

Theorem 16.9. *Let $\mathcal{L} \to \mathcal{Y}$ and $\mathcal{L} \to \mathcal{Z}$ be embeddings of pretopological stacks, and let $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ be a gluing (as in Theorem 16.4). Then:*

- **i**. *The natural functor*
$$\varrho\colon \mathrm{Hom}_{\mathsf{St}}(\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}, \mathcal{X}) \to \mathrm{Glue}(\mathcal{Y}, \mathcal{Z}, \mathcal{L}; \mathcal{X})$$
is fully faithful.
- **ii**. *If there is a chart $p\colon W \to \mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ such that the corresponding embedding $i\colon L \to Y$ is LTC (L and Y are invariant subspaces of W corresponding to $\mathcal{L}$ and $\mathcal{Y}$), then ϱ is an equivalence of groupoids for every topological stack $\mathcal{X}$.*

Proof. Pick a chart $p\colon W \to \mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ and let Y, Z and L be the invariant subspaces of W corresponding to $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{L}$, respectively. Using the description of maps coming out of a quotient stack (Section 3.5) the problem reduces to Theorem 16.2. Details left to the reader. □

Corollary 16.10. *If in part* (**ii**) *of Theorem 16.9 the pretopological stack $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ is actually topological, then it is unique (up to an equivalence that is unique up to 2-isomorphism). This is automatically the case if we take* **LF** *to be the class of locally cartesian maps (Example 13.1.***3***) or local homeomorphisms.*

Proof. The uniqueness follows from the universal property (Theorem 16.9). The last statement follows from Remark 16.8. ☐

Corollary 16.11. *Let $\mathcal{Y}$ and $\mathcal{Z}$ be topological stacks, and let $\mathcal{U} \subseteq \mathcal{Y}$ and $\mathcal{V} \subseteq \mathcal{Z}$ be open substacks. Suppose we have an equivalence $f\colon \mathcal{U} \xrightarrow{\sim} \mathcal{V}$. Then $\mathcal{Y}$ can be glued to $\mathcal{Z}$ along f to form a topological stack. The glued stack satisfies the universal property of push-out.*

In fact, as long as we are gluing along *open* substacks, any coherent gluing data can be used to glue topological stacks, and the resulting stack will again be topological (and will satisfy the relevant universal property). It should be clear how to formulate such a statement, but we will not do it here because it a bit messy and we will not be needing it later.

Theorem 16.12. *Consider the diagram topological stacks*

$$\begin{tikzcd} \mathcal{L} \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & \mathcal{Y} \\ \mathcal{Z} & \end{tikzcd}$$

in which i and j are embeddings.

- **i**. *If $\mathcal{Y}$ and $\mathcal{Z}$ are weak Deligne-Mumford, then so is $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$. Furthermore, $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ satisfies the universal property of push-out (page 56).*
- **ii**. *Assume i and j are locally-closed embeddings (that is, a closed embedding followed by an open embedding). If $\mathcal{Y}$ and $\mathcal{Z}$ are Deligne-Mumford, then so is $\mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$.*

Proof of part (**i**). Let **LF** be the class of local homeomorphisms. Then every embedding $i\colon L \to Y$ of topological spaces is LTC (Example 13.7.**6**) and neat (Example 16.6.**3**). Furthermore, if we glue local homeomorphisms along subspaces we obtain a local homeomorphism (see Remark 16.8). So the chart $X \to \mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$ constructed in Theorem 16.4 is étale.

Proof of part (**ii**). Pick an étale chart $X \to \mathcal{Y} \vee_{\mathcal{L}} \mathcal{Z}$. Let $x \in X$ be an arbitrary point. We have to find an open around x that is invariant under I_x and the action of I_x on it is mild. Let $L \subseteq X$ be the subspace of X corresponding to $\mathcal{L}$. If x is not in L the assertion is obvious. So assume x is in L. Let $Y, Z \subseteq X$ be the subspaces corresponding to $\mathcal{Y}$ and $\mathcal{Z}$. Since $\mathcal{Y}$ is Deligne-Mumford, there is a subset $U \subseteq Y$ containing x that is invariant under the action of I_x and such that the induced groupoid on U is isomorphic to the action groupoid $[I_x \times U \rightrightarrows U]$ of a mild action of I_x on U (Proposition 14.5). We can find $V \subseteq Z$ with the similar property. Since L is locally closed in both Y and Z, we may also assume that U and V are small enough so that $U \cap L$ and $V \cap L$ are closed in U and V, respectively. Let $U' = U \backslash \big(L \backslash (V \cap L)\big)$ and $V' = V \backslash \big(L \backslash (U \cap L)\big)$. These are open sets in U and V, respectively, and we have $U' \cap V' = U \cap V$. Furthermore, both U' and V' are I_x

invariant. Set $W = U' \cup V' \subseteq X$. It is easy to see that W is open in X, and the induced groupoid on W is isomorphic to the action groupoid $[I_x \times W \rightrightarrows W]$ of a mild action of I_x. $\square$

17. Elementary homotopy theory

In this section we begin developing homotopy theory of topological stacks. Recall that our definition of a topological stack depends on a choice of a class **LF** of local fibrations (Section 13.1). We require that **LF** is so that CW inclusions of finite CW complexes are LTC. This is the case for all the classes considers in Example 13.1 except for (**4**) and (**5**).

We use squiggly arrows $\rightsquigarrow$, instead of double arrows $\Rightarrow$, to denote 2-isomorphisms between points.

Definition 17.1. By a **pair** we mean a pair $(\mathcal{X}, \mathcal{A})$ of topological stacks and a morphism (not necessarily an embedding) $i \colon \mathcal{A} \to \mathcal{X}$ between them (we usualy drop i in the notation). When $\mathcal{A} = *$ is a point, we call this a **pointed stack**. A *morphism of pairs* $(\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ consists of two morphisms (f, f') and a 2-isomorphism as in the following 2-cell:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f'"] \arrow[d, "i"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{B} \arrow[d, "j"] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

We usually abuse notation and denote such a morphism by f. Given two morphisms $f, g \colon (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ of pairs, we define an **identification** (or *transformation*, *2-isomorphism*) from f to g to be a pair of transformations $f \overset{\varphi}{\Rightarrow} g$ and $f' \overset{\varphi'}{\Rightarrow} g'$ making the following 2-cell commute:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\varphi' \Downarrow"] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{X} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\varphi \Downarrow"] & \mathcal{Y} \end{tikzcd}$$

We usually abuse notation and denote such a transformation by φ.

For pointed stacks we use the (slightly different) notation $(\mathcal{X}, x)$, where $x \colon * \to \mathcal{X}$ is the given point.

A morphism of pointed stacks is a pair $(f, \phi) \colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$, where $f \colon \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks and $\phi \colon y \rightsquigarrow f(x)$ is an identification.

Definition 17.2. Let $f, g \colon (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ be maps of pairs as in the previous definition. A **homotopy** from f to g is a triple $(H, \epsilon_0, \epsilon_1)$ as follows:

- A map of pairs $H \colon (I \times \mathcal{X}, I \times \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$.
- A pair of identifications $f \overset{\epsilon_0}{\Rightarrow} H_0$ and $H_1 \overset{\epsilon_1}{\Rightarrow} g$. Here H_0 and H_1 stand for the morphisms of pairs $(\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ obtained by restricting H to $\{0\} \times \mathcal{X}$ and $\{1\} \times \mathcal{X}$, respectively.

We usually drop ϵ_0 and ϵ_1 in the notation. We denote the homotopy classes of maps of pairs $(\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ by $[(\mathcal{X}, \mathcal{A}), (\mathcal{Y}, \mathcal{B})]$. If $x, y \colon * \to \mathcal{X}$ are points in $\mathcal{X}$, a homotopy between x and y is called a **path** from x to y.

Remark 17.3. In view of the above definition, an identification between two maps of pairs can be regarded as a homotopy. For precisely, let $f, g \colon (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ be maps of pairs and $\varphi \colon f \Rightarrow g$ an identification. Then we have a homotopy $(H, \epsilon_0, \epsilon_1)$ from f to g where H is

$$(I \times \mathcal{X}, I \times \mathcal{A}) \xrightarrow{pr} (\mathcal{X}, \mathcal{A}) \xrightarrow{f} (\mathcal{Y}, \mathcal{B})$$

and $\epsilon_0 = \mathrm{id}$ and $\epsilon_1 = \varphi$.

When the pair $(\mathcal{X}, \mathcal{A})$ is reasonably well-behaved, we can compose homotopies. More precisely, we have the following result.

Lemma 17.4. *Let $(\mathcal{X}, \mathcal{A})$ be a pair. Assume $\mathcal{X}$ and $\mathcal{A}$ admit charts $X \to \mathcal{X}$ and $A \to \mathcal{A}$ such that the $t = 0$ inclusions $X \hookrightarrow I \times X$ and $A \hookrightarrow I \times A$ are LTC (e.g. X and A both CW complexes). Then, for any pair $(\mathcal{Y}, \mathcal{B})$ of topological stacks, homotopies between maps from $(\mathcal{X}, \mathcal{A})$ to $(\mathcal{Y}, \mathcal{B})$ can be composed in a natural way. In other words, given maps of pairs $f_1, f_2, f_3 \colon (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$, and homotopies H_{12} from f_1 to f_2 and H_{23} from f_2 to f_3, we can compose H_{12} and H_{23} in a natural way to obtain a homotopy H_{13} from f_1 to f_3.*

Proof. Follows from Theorem 16.9. $\square$

Definition 17.5. Let $(\mathcal{X}, x)$ be a pointed topological stack. We define $\pi_n(\mathcal{X}, x) = [(S^n, \bullet), (\mathcal{X}, x)]$ for $n \geq 0$.

As for topological spaces, there is a natural group structure on $\pi_n(\mathcal{X}, x)$ for every $n \geq 1$. The multiplication is defined as follows.

Let $f, g \colon (S^n, \bullet) \to (\mathcal{X}, x)$ be pointed maps. From the definition of a map of pairs (Definition 17.1), there is a natural identification $\alpha : f(\bullet) \rightsquigarrow g(\bullet)$. So, by Proposition 16.1, there is a map $f \vee g \colon S^n \vee S^n \to \mathcal{X}$ whose restrictions to each copy of S^n is naturally 'identified' with f and g. The restrictions of these identifications to the base points gives the following commutative diagram of identifications:

$$\begin{tikzcd} & x \arrow[dl, squiggly, no head] \arrow[dr, squiggly, no head] & \\ f(\bullet) \arrow[rr, squiggly, no head, "\alpha"] \arrow[dr, squiggly, no head] & & g(\bullet) \arrow[dl, squiggly, no head] \\ & (f \vee g)(\bullet) & \end{tikzcd}$$

In particular, we have an identification $x \rightsquigarrow (f \vee g)(\bullet)$, which makes $f \vee g \colon (S^n \vee S^n, \bullet) \to (\mathcal{X}, x)$ into a pointed map. It can be checked, using Proposition 16.1, that this construction respects homotopies of pointed maps (Definition 17.2). Precomposing $f \vee g$ with the $\frac{1}{2} + \frac{1}{2}$ map $S^n \to S^n \vee S^n$ gives rise to a pointed map $(S^n, \bullet) \to (\mathcal{X}, x)$, which we define to be the product fg of f and g in $\pi_n(\mathcal{X}, x)$.

Similarly, given three maps $f, g, h \colon (S^n, \bullet) \to (\mathcal{X}, x)$, we can construct a map $f \vee g \vee h \colon (S^n \vee S^n \vee S^n, \bullet) \to (\mathcal{X}, x)$, unique up to a unique 2-isomorphism, whose restriction to each copy of S^n is naturally identified with the corresponding map f, g or h.

Pre-composing this map with the $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}$ map $S^n \to S^n \vee S^n \vee S^n$ gives a pointed map $(S^n, \bullet) \to (\mathcal{X}, x)$ that is naturally identified with $(fg)h$. Similarly, pre-composing with the $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}$ map $S^n \to S^n \vee S^n \vee S^n$ gives $f(gh)$. Since the $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}$ and $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}$ maps are homotopic, they remain homotopic after composing with $f \vee g \vee h$, so we have $(fg)h = f(gh) \in \pi_n(\mathcal{X}, x)$. This verifies associativity. The other axioms of a group are verified in a similar fashion.

The homotopy groups of topological stacks as defined above are functorial with respect to pointed maps of topological stacks. That is, given a pointed map $(f, \phi)\colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$, we get an induced map on the homotopy groups

$$\pi_n(f, \phi)\colon \pi_n(\mathcal{X}, x) \to \pi_n(\mathcal{Y}, y).$$

Note that the above maps do depend on ϕ. We sometime denote this map by $\pi_n(f)$, $(f, \varphi)_*$, or f_*, if there is no fear of confusion.

Homotopy groups of topological stacks have all the natural properties of the homotopy groups of topological spaces. For instance, $\pi_n(\mathcal{X}, x)$ is abelian for $n \geq 2$, there is an action of $\pi_1(\mathcal{X}, x)$ on $\pi_n(\mathcal{X}, x)$, there is a Whitehead product which satisfies the Jacobi identity and so on. To see these, notice that in the classical case all these results reflect certain structures on the spheres S^n (e.g., $\pi_n(\mathcal{X}, x)$ is an abelian group for $n \geq 2$ because S^n is a homotopy abelian cogroup for $n \geq 2$). Therefore, arguments, similar to the one use above to prove the associativity of the multiplication on $\pi_n(\mathcal{X}, x)$, can be used to prove other properties of homotopy groups of stacks.

As pointed out in the previous paragraph, the homotopy groups of a topological stack behave pretty much like the homotopy groups of topological spaces. There is, however, more structure on the homotopy groups of a topological stack. We discuss the case of fundamental group in detail. The higher cases have been briefly touched upon in Section 19.2.

For a pointed topological stack $(\mathcal{X}, x)$ there is natural group homomorphism

$$\omega_x\colon I_x \to \pi_1(\mathcal{X}, x)$$

(see Remark 17.3). For this reason, and also to be consistent with the case of higher homotopy groups (see Section 19.2), we sometimes refer to I_x as the **inertial fundamental group** of $(\mathcal{X}, x)$. Recall that we have a natural isomorphism

$$I_x \cong \{\alpha \mid \alpha\colon x \rightsquigarrow x \text{ self-transformation}\}.$$

The maps ω_x are functorial with respect to pointed maps. That is, given a pointed map $(f, \phi)\colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$, we get an induced map $(f, \phi)_*\colon I_x \to I_y$, and the following square commutes:

$$\begin{tikzcd} I_x \arrow[r, "\omega_x"] \arrow[d, "(f,\phi)_*"'] & \pi_1(\mathcal{X}, x) \arrow[d, "\pi_1(f,\phi)"] \\ I_y \arrow[r, "\omega_y"'] & \pi_1(\mathcal{Y}, y) \end{tikzcd}$$

We will use inertial fundamental groups in the study of covering spaces of topological stacks.

18. Covering theory for topological stacks

We develop a Galois theory of covering spaces for a topological stack. In particular, we show that, under appropriate local assumptions, there is natural correspondence between subgroups of the fundamental group $\pi_1(\mathcal{X}, x)$ of a pointed topological stack $(\mathcal{X}, x)$ and the (isomorphism classes of) pointed covering spaces of $(\mathcal{X}, x)$. We then explain the role of inertial fundamental groups as bookkeeping devices which keep track of the stacky structure of covering spaces. Similar results have been proven in the algebraic setting in [No1], and most of the proofs can be easily adopted to the topological setting.

We fix a class **LF** of local fibrations as in Section 17.

18.1. **Connectedness conditions.** In this subsection we generalize several notions of connectedness topological stacks.

In what follows, whenever we talk about connected components, we can assume our class **LF** of local fibrations is arbitrary. So our discussion will be valid for pretopological stacks too. However, when we talk about *path* components, we have to assume **LF** is such that the inclusion $\{0\} \hookrightarrow I$ is LTC. This is the case for every example in Example 13.1 except for (**4**) and (**5**).

Definition 18.1. Let $\mathcal{X}$ be a pretopological stack. We say $\mathcal{X}$ is **connected** if it has no proper open-closed substack. We say $\mathcal{X}$ is **path connected**, if for every two points x and y in $\mathcal{X}$, there is a path from x to y (Definition 17.2).

Definition 18.2. Let $\mathcal{X}$ be a pretopological stack. We say $\mathcal{X}$ is **locally connected** (resp., **locally path connected**, **semilocally 1-connected**), if there is a chart $X \to \mathcal{X}$ such that X is so.

This definition agrees with the usual definition when $\mathcal{X}$ is a topological space. This is because of the following lemma.

Lemma 18.3. *Let $f \colon Y \to X$ be an epimorphism of topological spaces. Assume Y is locally connected (resp., locally path connected, semilocally 1-connected). Then so is X.*

Proof of locally connected. Being locally connected is a local condition, so we may assume f has a section $s \colon X \to Y$. Recall that X is locally connected if an only if for every open subset $U \subseteq X$ every connected component of U is open (hence also closed). So it is enough to show that every connected component of X is open; same argument will work for any open subset U. Let $Y = \coprod Y_i$ be the decomposition of Y into connected components. So every Y_i is open-closed. For each i, $X_i := s^{-1}(Y_i)$ is open-closed (possibly empty). It is also connected because, if X_i is not empty, f surjects Y_i onto X_i (note that $f(Y_i) \subseteq X_i$ since Y_i is connected and X_i is open-closed). Also note that, if $i \neq j$, then X_i and X_j are disjoint. This shows that the connected components of X are exactly those X_i that are non-empty. So all the connected components of X are open.

Proof of locally path connected. Similar to the previous case.

Proof of semilocally 1-connected. Let $x \in X$ be an arbitrary point, and assume U is an open neighborhood of X over which f has a section $s \colon U \to Y$. Let V be an open neighborhood of $y = s(x)$ such that $\pi_1(V, y) \to \pi_1(Y, y)$ is trivial. It is easily seen that $s^{-1}(V) \subseteq X$ is an open neighborhood of x with the similar property. $\square$

It is easy to see that, $\mathcal{X}$ is locally (path) connected if and only if for every map $f\colon W \to \mathcal{X}$ from a topological space W to $\mathcal{X}$, and every point $w \in W$, there is an open neighborhood $U \subseteq W$ of w in W such that $f|_U$ factors through a map $V \to \mathcal{X}$, where V is a locally (path) connected topological space.

Similarly, $\mathcal{X}$ is semi-locally 1-connected if and only if for every map $f\colon W \to \mathcal{X}$ from a topological space W to $\mathcal{X}$, and every point $w \in W$, there is an open neighborhood $U \subseteq W$ of w in W such that induced map on fundamental groups $\pi_1(U, w) \to \pi_1(\mathcal{X}, f(w))$ is the zero map.

Lemma 18.4.

- **i**. *Let $\mathcal{X}$ be a pretopological stack. Then, $\mathcal{X}$ is connected if and only if X_{mod} is so. If $\mathcal{X}$ is path connected, then so is X_{mod}.*
- **ii**. *Let $\mathcal{X}$ be a pretopological stack. If $\mathcal{X}$ is locally connected (resp., locally path connected), then so is X_{mod}.*

Proof of part (**i**). Obvious.

Proof of part (**ii**). We only prove the locally connected case. The locally path connected case is proved analogously.

Let $X \to \mathcal{X}$ be a locally connected chart for $\mathcal{X}$. Then $f\colon X \to X_{mod}$ is a quotient map (Example 4.13). So we have to show that, if $f\colon X \to Y$ is a quotient map of topological spaces and X is locally connected, then so is Y. We show that, for every open $U \subseteq Y$, the connected components of U are open. Since the restriction of f over U is again a quotient map, we are reduced to the case $U = Y$. Let Y_0 be a connected component of Y. We have to show that $f^{-1}(Y_0)$ is open in X. But $f^{-1}(Y_0)$ is a union of connected components of X. The claim follows. □

It is possible to have a topological stack that is not locally connected and not path connected but whose coarse moduli space is path connected and locally path connected (see Example 19.1).

Definition 18.5. Let $\mathcal{X}$ be a topological stack (see the assumption on **LF** at the beginning of this subsection), and let x be a point in $\mathcal{X}$. The (**path**) **component** of x is defined to be the union (Section 4.2) of all (path) connected embedded substacks of $\mathcal{X}$ containing x.

It is an easy exercise that each component is connected. Similarly, each path component is path connected (Proposition 16.1 is needed for this). The connected components of $\mathcal{X}$ correspond exactly to connected components of X_{mod} (in the obvious way). The path components of $\mathcal{X}$ give rise to a partitioning of X_{mod} that is finer than (or equal to) the partitioning of X_{mod} by its path components.

Lemma 18.6. *Let $\mathcal{X}$ be a topological stack and $p\colon X \to \mathcal{X}$ a chart for it. Then, for every (path) component $\mathcal{X}_0$ of $\mathcal{X}$, the corresponding invariant subset $X_0 \subseteq X$ is a union of (path) components of X. In particular, if $\mathcal{X}$ is locally (path) connected, then every (path) component of $\mathcal{X}$ is an open-closed substack.*

Proof. Let $A \subseteq X$ be a (path) component of X that is not a subset of X but $X_0 \cap A \neq \emptyset$. Then it is easy to see that $\mathcal{O}(X_0 \cup A)$ is an invariant subspace of X whose corresponding embedded substack in $\mathcal{X}$ is (path) connected, and strictly bigger than $\mathcal{X}_0$. This is a contradiction. □

Corollary 18.7. *Let $\mathcal{X}$ be a connected, locally path connected topological stack. Then $\mathcal{X}$ is path connected.*

The following lemmas will be used later.

Lemma 18.8. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a surjective local homeomorphism of pretopological stacks. Then $\mathcal{X}$ is locally (path) connected if and only if $\mathcal{Y}$ is so.*

Proof. If $\mathcal{X}$ is locally (path) connected, the pull back of locally (path) connected chart for $\mathcal{X}$ is a locally (path) connected chart for $\mathcal{Y}$. Conversely, if $Y \to \mathcal{Y}$ is a locally (path) connected chart for $\mathcal{Y}$, the composition $Y \to \mathcal{Y} \to \mathcal{X}$ will be a locally (path) connected chart for $\mathcal{X}$. (Note that every surjective local homeomorphism is an epimorphism.) $\square$

Lemma 18.9. *Let $\mathcal{X}$ be a locally (path) connected weak Deligne-Mumford topological stack, and let $p\colon X \to \mathcal{X}$ be an étale chart for it. Then X is locally (path) connected. Same statement is true with semilocally 1-connected.*

Proof. Let $q\colon X' \to \mathcal{X}$ be a chart such that X' is locally (path) connected. Then $Y := X \times_{\mathcal{X}} X'$ is also locally (path) connected, because the projection map $X \times_{\mathcal{X}} X' \to X'$ is a local homeomorphism and X' is locally connected. The map $Y \to X$ is now an epimorphism from a locally (path) connected space to X. This implies that X is also locally (path) connected (Lemma 18.3). $\square$

18.2. **Galois theory of covering spaces of a topological stack.** We begin with recalling the definition of a covering map of pretopological stacks.

Definition 18.10. A map $\mathcal{Y} \to \mathcal{X}$ of pretopological stacks is a **covering map**, if it is representable, and for every topological space W and every map $W \to \mathcal{X}$, the base extension $W \times_{\mathcal{X}} \mathcal{Y} \to W$ is a covering map of topological spaces.

Proposition 18.11. *Let $p\colon \mathcal{Y} \to \mathcal{X}$ be a covering map of pretopological stacks. Then the diagonal map $\Delta\colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is an open-closed embedding.*

Proof. Let $X \to \mathcal{X}$ be a chart for $\mathcal{X}$, and let $Y \to \mathcal{Y}$ be the pull back chart for $\mathcal{Y}$. The base extension $q\colon Y \to X$ of p is a covering map. We have a 2-cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\Delta"'] & \mathcal{Y} \arrow[d, "\Delta"] \\ Y \times_X Y \arrow[r] & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \end{tikzcd}$$

Since $q\colon Y \to X$ is a covering map of topological spaces, the left vertical map is an open-closed embedding. So we have shown that the base extension of $\Delta\colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ along the epimorphism $Y \times_X Y \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is an open-closed embedding. So $\Delta\colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ itself is also an open-closed embedding. $\square$

Using the above proposition, we can prove the uniqueness of liftings for covering maps.

Lemma 18.12. *Let $p\colon \mathcal{Y} \to \mathcal{X}$ be a covering map of pretopological stacks. Let $\mathcal{T}$ be a connected pretopological stack, and let t be a point in $\mathcal{T}$. Let $f, g\colon \mathcal{T} \to \mathcal{Y}$ be maps, and let $\Phi\colon p \circ f \Rightarrow p \circ g$ be a 2-isomorphism. Assume also $f(t)$ and $g(t)$*

are equivalent points in $\mathcal{Y}$, *and fix a 2-isomorphism* $\psi\colon f(t) \rightsquigarrow g(t)$. *Then, there is a unique 2-isomorphism* $\Psi\colon f \Rightarrow g$ *such that its restriction to* t *is equal to* ψ *and* $p \circ \Psi = \Phi$.

Proof. Consider the map $(f,g,\Phi)\colon \mathcal{T} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$. Form the 2-fiber product

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d, "d"'] & \mathcal{Y} \arrow[d, "\Delta"] \\ \mathcal{T} \arrow[r, "(f,g,\Phi)"'] \arrow[ur, dotted] & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \end{tikzcd}$$

The assertion is equivalent to saying that, $(f,g,\Phi)\colon \mathcal{T} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ can be lifted to $\mathcal{Y}$ as a pointed map. Note that this is a 2-cartesian diagram of pointed stacks, where we take $f(t)$ for the base point of $\mathcal{Y}$, and $\big(f(t), f(t), id\big)$ for the base point of $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$). The base point of $\mathcal{Z}$ is obtained from ψ.

The map $d\colon \mathcal{Z} \to \mathcal{T}$ is a base extension of Δ, so, by Proposition 18.11, it is an open-closed embedding. Therefore, d is an equivalence of pointed stacks. Composing an inverse of d with the projection $\mathcal{Z} \to \mathcal{Y}$ gives us the desired lift. $\square$

Definition 18.13. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map of stacks, and $x \in \mathcal{X}$ a point in $\mathcal{X}$. We define the **fiber** $\mathcal{Y}_x$ of $\mathcal{Y}$ over x to be the following groupoid:

- $\mathrm{Ob}(\mathcal{Y}_x) = \left\{ \begin{array}{ccl} (y,\phi) & | & y\colon * \to \mathcal{Y},\ \text{a point in } \mathcal{Y} \\ & & \phi\colon x \rightsquigarrow f(y),\ \text{an identification} \end{array} \right\}$
- $\mathrm{Mor}_{\mathcal{Y}_x}\big((y,\phi),(y',\phi')\big) = \{\beta\colon y \rightsquigarrow y' \mid \phi \cdot f(\beta) = \phi'\}$

In other words, $\mathcal{Y}_x$ is the groupoid of points in the stack $* \times_{\mathcal{X}} \mathcal{Y}$. When f is representable, this groupoid is equivalent to a set.

Now, we introduce the Galois category $\mathsf{Cov}_{\mathcal{X}}$ of a path connected topological stack $\mathcal{X}$. The Galois category $\mathsf{Cov}_{\mathcal{X}}$ of $\mathcal{X}$ is defined as follows:

- $\mathrm{Ob}(\mathsf{Cov}_{\mathcal{X}}) = \left\{ \begin{array}{ccr} (\mathcal{Y},f) & | & \mathcal{Y},\ \text{a topological stack;} \\ & & f\colon \mathcal{Y} \to \mathcal{X},\ \text{a covering space.} \end{array} \right\}$
- $\mathrm{Mor}_{\mathsf{Cov}_{\mathcal{X}}}\big((\mathcal{Y},f),(\mathcal{Z},g)\big) = \left\{ \begin{array}{ccr} (a,\Phi) & | & a\colon \mathcal{Y} \to \mathcal{Z}\ \ \text{a morphism;} \\ & & \Phi\colon f \Rightarrow g \circ a,\ \text{a 2-morphism.} \end{array} \right\}_{/\sim}$

Here, $\sim$ is defined by

$$(a,\Phi) \sim (b,\Psi) \text{ if } \exists \Gamma\colon a \Rightarrow b \text{ such that } \Phi \cdot g(\Gamma) = \Psi.$$

The fiber functor $F_x\colon \mathsf{Cov}_{\mathcal{X}} \to (\pi_1(\mathcal{X},x){-}\mathsf{Set})$ is defined by $F_x(\mathcal{Y}) := \pi_0(\mathcal{Y}_x)$, where $\pi_0(\mathcal{Y}_x)$ stands for the set of isomorphism classes of $\mathcal{Y}_x$ (Definition 18.13). Note that, since covering maps are representable, $\mathcal{Y}_x$ is equivalent to a set. Also note that, if we have a pointed map $(\mathcal{Y},y) \to (\mathcal{X},x)$, then $F_x(\mathcal{Y})$ is naturally pointed.

The action of $\pi_1(\mathcal{X},x)$ on $F_x(\mathcal{Y})$ is defined as follows. Let $\gamma\colon I \to \mathcal{X}$ be a map representing a loop at x (in particular, we are given identifications $\epsilon_0\colon x \rightsquigarrow \gamma(0)$ and $\epsilon_1\colon \gamma(1) \rightsquigarrow x$). Since $\mathcal{Y} \to \mathcal{X}$ is a covering map, the base extension $E = I \times_{\mathcal{X}} \mathcal{Y}$ is isomorphic to a disjoint union of copies I. There are natural bijections $\varepsilon_0\colon E_0 \xrightarrow{\sim} F_x(\mathcal{Y})$ and $\varepsilon_1\colon F_x(\mathcal{Y}) \xrightarrow{\sim} E_1$ corresponding to ϵ_0 and ϵ_1, where E_0 and E_1 are fibers of E over 0 and 1, respectively. Since E_0 and E_1 are canonically identified, we get an action $\varepsilon_0 \circ \varepsilon_1 \colon F_x(\mathcal{Y}) \to F_x(\mathcal{Y})$. The same base extension trick, this time applied to $I \times I$, shows that this action is independent of the pointed homotopy class of γ. It is also easy to check that this action respects composition of loops, hence induces a group action of $\pi_1(\mathcal{X}, x)$ on $F_x(\mathcal{Y})$. This action can indeed be jazzed up to give a local system on the fundamental groupoid $\Pi_1(\mathcal{X})$ with fibers $F_x(\mathcal{Y})$, $x \in \mathcal{X}$.

An interesting special case is when γ comes from I_x. More precisely, let $\gamma \in I_x$ be an element in the inertia group at x, and consider the loop $\omega_x(\gamma) \colon I \to \mathcal{X}$ that is given by the constant map (i.e. the map that factors through x on the nose) together with the identifications $\epsilon_0 = id \colon x \rightsquigarrow x$ and $\epsilon_1 = \gamma \colon x \rightsquigarrow x$ (Definition 17.2). It is easy to check that the action of γ on $F_x(\mathcal{Y})$ sends (the class of) (y, ϕ) in F_x to (the class of) $(y, \phi \cdot \gamma)$, where $\cdot$ stands for composition of identifications.

Lemma 18.14. *Let $p \colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$ be a pointed covering map of topological stacks. Let $\alpha \colon (S^1, \bullet) \to (\mathcal{X}, x)$ be a loop. Then the following are equivalent:*

- **i**. *The action of α on $F_x(\mathcal{Y})$ leaves y invariant.*
- **ii**. *There is a (necessarily unique) lift $\tilde{\alpha} \colon (S, *) \to (\mathcal{Y}, y)$.*
- **iii**. *The class of α in $\pi_1(\mathcal{X}, x)$ is in $p_*(\pi_1(\mathcal{Y}, y))$.*

Proof of (**i**) $\Rightarrow$ (**ii**). Notations being as in the preceding paragraphs, let $y_0 \in E_0$ and $y_1 \in E_1$ be the points corresponding to y (via ϵ_0 and ϵ_1, respectively). Since y is invariant under the action of $\pi_1(\mathcal{X}, x)$, y_0 and y_1 lie in the same "layer" in E, that is, there is a section $s \colon I \to E$ whose end points are y_0 and y_1. Composing s with the natural map $E \to \mathcal{Y}$ we obtain a path whose end points are canonically identified with y. By Proposition 16.1, this gives a loop $\tilde{\alpha} \colon (S, \bullet) \to (\mathcal{Y}, y)$, which is the desired lift.

Proof of (**ii**) $\Rightarrow$ (**iii**). Obvious.

Proof of (**iii**) $\Rightarrow$ (**i**). Since the action is independent of the choice the (pointed) homotopy class of the loop, we may assume that α lifts to $(\mathcal{Y}, y)$. Now run the proof of (**i**) $\Rightarrow$ (**ii**) backwards. $\square$

Proposition 18.15. *Let $p \colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$ be a pointed covering map. Then the induced map $\pi_1(\mathcal{Y}, y) \to \pi_1(\mathcal{X}, x)$ is injective.*

Proof. Let $\alpha, \beta \colon (S, \bullet) \to (\mathcal{Y}, y)$ be loops whose images in $\pi_1(\mathcal{X}, x)$ are equal. Then, there exists a pointed homotopy $H \colon (I \times S^1, I \times \{\bullet\}) \to (\mathcal{X}, x)$ between $p \circ \alpha$ and $p \circ \beta$. Let $A = (\{0\} \times S^1) \cup (I \times \{\bullet\}) \cup (\{1\} \times S^1) \subset I \times S^1$. By Proposition 16.1 (or Theorem 16.2), there is a natural map $f \colon A \to \mathcal{Y}$ whose restrictions to $\{0\} \times S^1$, $\{1\} \times S^1$ and $I \times \{\bullet\}$ are naturally identified with α, β, and the constant map. By Theorem 16.2, the composition $p \circ f \colon A \to \mathcal{X}$ is naturally identified with $H|_A$ (to see this, note that such an identification exists on each of the three pieces A is made of, and then use Theorem 16.2 to show that they glue to an identification defined on A).

Set $Z = (I \times S^1) \times_{\mathcal{X}} \mathcal{Y}$. Consider the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, shift left, "q"] & \mathcal{Y} \arrow[d, "p"] \\
I \times S^1 \arrow[u, dotted, shift left, "s"] \arrow[ur, dotted, "F"'] \arrow[r, "H"'] & \mathcal{X}
\end{tikzcd}$$

Note that Z is a topological space and q is a covering map. The map f gives rise to a partial section $\bar{f}\colon A \to Z$ of q defined on A. Since q is a covering map, $\bar{f}$ extends to a section $s\colon I \times S^1 \to Z$. The composite $F = g \circ s$ is the desired homotopy between α and β. □

Lemma 18.16. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a map of topological stacks. Let y be a point of $\mathcal{Y}$ and $x = f(y)$ its image in $\mathcal{X}$. Consider the following commutative diagram, obtained from the functoriality of the inertial fundamental groups:*

$$\begin{array}{ccc} I_y & \xrightarrow{\omega_y} & \pi_1(\mathcal{Y}, y) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \pi_1(f)} \\ I_x & \xrightarrow[\omega_x]{} & \pi_1(\mathcal{X}, x) \end{array}$$

If f is a covering map, then this diagram is cartesian.

Proof. Let γ be in I_x. We want to show that, if $\omega_x(\gamma)$ is in the image of $\pi_1(f)$, then there exists a unique $\alpha \in I_y$ such that $f_*(\alpha) = \gamma$. The uniqueness is obvious, because $f_*\colon I_y \to I_x$ is injective (use Proposition 4.3.**ii**). On the other hand, $\omega_x(\gamma)$ being in the image of $\pi_1(f)$ exactly means that, under the action of γ on $F_x(\mathcal{Y})$, the point (y, id) remains invariant (Lemma 18.14). That means, $(y, id) \sim (y, \gamma)$, where $\sim$ means an isomorphism in the groupoid $\mathcal{Y}_x$ (see Definition 18.13), which in this case is indeed equivalent to a set. Therefore, there exists $\beta \in I_y$ such that $f(\beta) = \gamma$. □

Lemma 18.17. *Let $p\colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$ be a pointed covering map of path connected topological stacks. If the induced map $\pi_1(\mathcal{Y}, y) \to \pi_1(\mathcal{X}, x)$ is an isomorphism, then p is an equivalence.*

Proof. It is enough to show that the fiber $F_x(\mathcal{Y})$ is the set with one element, that element being the class of (y, ϕ), where $\phi\colon x \rightsquigarrow p(y)$ is part of the data of a pointed map. (More precisely, if this is proven, then path connectedness implies that the fiber over every point $x' \in \mathcal{X}$ is a set with one element. Therefore, for any map $X \to \mathcal{X}$ from a topological space X to $\mathcal{X}$, the base extension $Y \to X$ of p will be a covering map of degree one, hence an isomorphism. In particular, the base extension of p along a chart $X \to \mathcal{X}$ is an isomorphism. Hence p must be an equivalence by Lemma 6.1.) Assume (y', ϕ') is another point in $F_x(\mathcal{Y})$. Since $\mathcal{Y}$ is path connected (Corollary 18.7 and Lemma 18.8), we can find a path γ connecting y and y'. The image of γ in $\mathcal{X}$ is a map $I \to \mathcal{X}$ such that the images of the end points are canonically equivalent to x. So, by Proposition 3.4, we obtain a loop $\alpha\colon (S^1, \bullet) \to (\mathcal{X}, x)$. By hypothesis, α is in $p_*\big(\pi_1(\mathcal{Y}, y)\big)$. By Lemma 18.14, we can lift α to a loop $\tilde{\alpha}\colon (S^1, \bullet) \to (\mathcal{Y}, y)$ at y. This loop gives rise to a map $\gamma'\colon I \to \mathcal{Y}$, sending both 0 and 1 to (points equivalent to) y. Now, we have two maps $\gamma, \gamma'\colon I \to \mathcal{Y}$, both lifting α, and both sending 0 to (points equivalent to) y. So, by Uniqueness Lemma 18.12, γ and γ' are equivalent. Therefore $\gamma(1)$ and $\gamma'(1)$ are also equivalent. But the first one is (equivalent to) y and the second one is (equivalent to) y'. Hence, we obtain an identification $\rho\colon y \rightsquigarrow y'$.

We are not done yet, because existence of ρ is not enough to guarantee that $(y, \phi) \sim (y', \phi')$. We have to verify that $\phi \cdot p(\rho) = \phi'$. In fact, this is not necessarily true. However, we can adjust ρ as follows. Set $\beta = \phi^{-1} \cdot \phi' \cdot p(\rho)^{-1} \in I_{p(y)}$. By

Lemma 18.16, there exists $\tilde{\beta} \in I_x$ such that $f(\tilde{\beta}) = \beta$. Now, if we replace ρ by $\sigma = \tilde{\beta} \cdot \rho$ we have our desired identification $\sigma \colon y \rightsquigarrow y'$ (i.e $\phi \cdot p(\sigma) = \phi'$). $\square$

Proposition 18.18 (The general lifting lemma). *Let $(\mathcal{Y}, y)$ and $(\mathcal{X}, x)$ be pointed connected locally path connected topological stacks, and let $p \colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$ be a pointed covering map. Suppose $(\mathcal{T}, t)$ is a pointed connected locally path topological stack, and let $f \colon (\mathcal{T}, t) \to (\mathcal{X}, x)$ be a pointed map. Then, f can be lifted to a pointed map $\tilde{f} \colon (\mathcal{T}, t) \to (\mathcal{Y}, y)$ if and only if $f_*\big(\pi_1(\mathcal{T}, t)\big) \subseteq p_*\big(\pi_1(\mathcal{Y}, y)\big)$. Furthermore, if such a lifting exists, it is unique. (Of course, the terms 'lifting' and 'uniquness' are to be interpreted as 'up to unique 2-isomorphism'.)*

Proof. One implication it trivial. Assume now that $f_*(\pi_1(\mathcal{T}, t)) \subseteq p_*(\pi_1(\mathcal{Y}, y))$. Let $(\mathcal{Z}, z)$ be the connected component of the 2-fiber product $\mathcal{T} \times_{\mathcal{X}} \mathcal{Y}$ containing the canonical base point z. The map $q \colon (\mathcal{Z}, z) \to (\mathcal{T}, t)$ is a pointed covering map. It follows from the hypothesis that the induced map $\pi_1(\mathcal{Z}, z) \to \pi_1(\mathcal{T}, t)$ is surjective (Lemma 18.14), hence an isomorphism (Proposition 18.15). Therefore, by Lemma 18.17, q is an equivalence. Pick a (pointed) inverse for q and compose it with $(\mathcal{Z}, z) \to (\mathcal{Y}, y)$ to obtain the desired lifting. $\square$

The main theorem of this section is

Theorem 18.19. *Let $(\mathcal{X}, x)$ be a pointed connected topological stack. Assume $\mathcal{X}$ is locally path connected and semilocally 1-connected (Definition 18.2). Then, the functor $F_x \colon \mathsf{Cov}_{\mathcal{X}} \to (\pi_1(\mathcal{X}, x) - \mathsf{Set})$ is an equivalence of categories.*

Before proving Theorem 18.19, we remark that, under the correspondence stated in the theorem, the connected covering spaces correspond to transitive $\pi_1(\mathcal{X}, x)$-sets. Hence, we have the following

Corollary 18.20. *Let $(\mathcal{X}, x)$ be a pointed connected topological stack. Assume $\mathcal{X}$ is locally path connected and semilocally 1-connected. Then, there is a natural bijection between the subgroups $H \subseteq \pi_1(\mathcal{X}, x)$ and the (isomorphism classes of) pointed connected covering maps $p \colon (\mathcal{Y}, y) \to (\mathcal{X}, x)$. In particular, $\mathcal{X}$ has a universal cover.*

The group H in the above corollary is nothing but $p_*\big(\pi_1(\mathcal{Y}, y)\big)$ (which is isomorphic to $\pi_1(\mathcal{Y}, y)$ by Proposition 18.15).

Before proving Theorem 18.19, we need some preliminaries.

Definition 18.21. Let $(\mathcal{X}, x)$ be a topological stack, and let A be a $\pi_1(\mathcal{X}, x)$-set. Let $f \colon W \to \mathcal{X}$ be a map from a path connected topological spaces W (unpointed) such that the induced map on fundamental groups is trivial. By an *A-equivalence class of a path from x to W (via f)* we mean a triple (w, γ, a), where w is a point in W, a a point in A, and γ is a path from x to $f(w)$, modulo the following equivalence relation:

> $(w, \gamma, a) \sim (w', \gamma', a')$ if there is a path α in X from w to w' such that the action of $\gamma f(\alpha) \gamma'^{-1} \in \pi_1(\mathcal{X}, x)$ sends a to a'.

Note that the choice of α is immaterial since f induces the trivial map on fundamental groups. Also note that if $f, g \colon W \to \mathcal{X}$ are 2-isomorphic maps, then the A-equivalence classes of paths from x to W via f are in natural bijection with the A-equivalence classes of paths from x to W via g.

Proof of Theorem 18.19. Fully faithfulness follows from Proposition 18.18. We have to prove essential surjectivity.

Take a $\pi_1(\mathcal{X},x)$-Set A. We construct the covering space $\mathcal{Y} \to \mathcal{X}$ associated to A as a sheaf (of sets) over the comma category $\mathsf{Top}_{\mathcal{X}}$ of topological spaces over $\mathcal{X}$. Recall that the comma category $\mathsf{Top}_{\mathcal{X}}$ is defined in a similar way to $\mathsf{Cov}_{\mathcal{X}}$ (see page 64), but we take the objects to be *all* maps $X \to \mathcal{X}$, where X is a topological space.

Define a presheaf $\mathcal{F}$ on $\mathsf{Top}_{\mathcal{X}}$ as follows. Let $f\colon W \to \mathcal{X}$ be an object in $\mathsf{Top}_{\mathcal{X}}$. Suppose W is connected and locally path connected, and that f induces the trivial map on the fundamental groups. Define

$$\mathcal{F}(f) = \{\ A\text{-equiv. classes of paths from } x \text{ to } W.\}$$

For an arbitrary object $g\colon V \to \mathcal{X}$ in $\mathsf{Top}_{\mathcal{X}}$, assume there exists an object $W \to \mathcal{X}$ as above and a morphism $\varphi\colon g \to f$ in $\mathsf{Top}_{\mathcal{X}}$. Then we set $\mathcal{F}(g) := \mathcal{F}(f)$. Observe that, for another choice of $f'\colon W' \to \mathcal{X}$ and $\varphi'\colon g \to f'$, there is a canonical bijection between $\mathcal{F}(f)$ and $\mathcal{F}(f')$ (exercise). So $\mathcal{F}(g)$ is well-defined. If such a W does not exists, we set $\mathcal{F}(V) = \emptyset$.

Now, define $\mathcal{Y}$ to be the sheafification $\mathcal{F}^a$. We claim that $\mathcal{Y}$, viewed as a stack over $\mathcal{X}$, is the desired covering stack. To see this, observe that, if $f\colon W \to \mathcal{X}$ is an object in $\mathsf{Top}_{\mathcal{X}}$ such that W is connected and locally path connected, and f induces the trivial map on the fundamental groups, then the restriction of $\mathcal{F}^a$ to Top_W is a constant sheaf. In fact, upon fixing a point w in W and a path γ in $\mathcal{X}$ from x to $f(w)$, this sheaf becomes canonically isomorphic to the constant sheaf associated to A.

The above observation translates as saying that, for every such $f\colon W \to \mathcal{X}$, the fiber product $W \times_{\mathcal{X}} \mathcal{Y}$ is equivalent (as a stack) to the topological space $W \times A$. Since $\mathcal{X}$ is locally path connected and semilocally 1-connected, we can find a family of map $W_i \to \mathcal{X}$ with the above property such that the induced map $\coprod W_i \to \mathcal{X}$ is an epimorphism. Lemma 6.3 now implies that $\mathcal{Y} \to \mathcal{X}$ is representable. It is also a covering map by Definition 4.5 (note that being a covering map is invariant under base extension and local on the target).

It is straightforward to check that the fiber of $\mathcal{Y} \to \mathcal{X}$ is canonically isomorphic to A as a $\pi_1(\mathcal{X},x)$-set. $\square$

Remark 18.22. The Galois category $\mathsf{Cov}_{\mathcal{X}}$ is in fact defined for every connected *pretopological* stack $\mathcal{X}$. Also, for a base point $x \in \mathcal{X}$, there is a fiber functor $F'_x\colon \mathsf{Cov}_{\mathcal{X}} \to \mathsf{Set}$. If we define $\pi'_1(\mathcal{X},x)$ to be the group of automorphisms of the functor F'_x, then it can be shown that $\mathsf{Cov}_{\mathcal{X}}$ is equivalent to $\pi'_1(\mathcal{X},x) - \mathsf{Set}$. When $\mathcal{X}$ is topological, there is a natural group homomorphism $\pi_1(\mathcal{X},x) \to \pi'_1(\mathcal{X},x)$. This map is an isomorphism if $\mathcal{X}$ is locally path connected and semilocally 1-connected.

18.3. **Role of the inertial fundamental groups.** Having the apparatus of Galois categories at hand, we can extend the results of [No1] to the topological setting. The role played by inertial fundamental groups is that they control the stacky structure of the covering spaces (see Lemma 18.16 and Theorem 18.24). [9]

Proposition 18.23. *Let $(\mathcal{X},x)$ be a pointed topological stack, and let $\tilde{\mathcal{X}}$ be a universal cover for $\mathcal{X}$. Then, for any point $x_0 \in \tilde{\mathcal{X}}$ lying above x, we have an isomorphism $I_{x_0} \cong \ker \omega_x$.*

[9] Inertial fundamental groups can also be used to compute the fundamental group of the coarse moduli space ([No2]).

Proof. Follows from Lemma 18.16. □

Theorem 18.24. *Let $\mathcal{X}$ be a connected locally path connected semilocally 1-connected topological stack, and let $\tilde{\mathcal{X}}$ be its universal cover. Then the following conditions are equivalent:*

i. *The maps $\omega_x \colon I_x \to \pi_1(\mathcal{X}, x)$ are injective for every point x.*
ii. *$\tilde{\mathcal{X}}$ is a quasitopological space;*
iii. *$\mathcal{X}$ is the quotient stack of the action of a discrete group on a quasitopological space.*

Furthermore, if $\mathcal{X}$ is a Deligne-Mumford topological stack, we can drop 'quasi' in (**ii**) *and* (**iii**)*.*

Proof of (**i**) $\Rightarrow$ (**ii**). This follows from Proposition 18.23.

Proof of (**ii**) $\Rightarrow$ (**iii**). Let $R = \tilde{\mathcal{X}} \times_{\mathcal{X}} \tilde{\mathcal{X}}$, and consider the groupoid $[R \rightrightarrows \tilde{\mathcal{X}}]$ (in the category of sheaves of sets over Top). The source and target maps are covering maps, and $\tilde{\mathcal{X}}$ is connected and simply connected. So R is a disjoint union of copies of $\tilde{\mathcal{X}}$ (indexed by $\pi_1\mathcal{X}$), each of which mapping homeomorphically to $\tilde{X}$ via the source and target maps. Lemma 7.7 is easily adapted to this situation and it implies that $[R \rightrightarrows \tilde{\mathcal{X}}]$ is the action groupoid of an action of $\pi_1\mathcal{X}$ on $\tilde{\mathcal{X}}$. (Having chosen a base point in $\tilde{\mathcal{X}}$ this action of $\pi_1\mathcal{X}$ on $\tilde{\mathcal{X}}$ could also be recovered from the general Lifting Lemma 18.18.)

Proof of (**iii**) $\Rightarrow$ (**i**). In this case, $\mathcal{X}$ has a covering stack in which all the inertial fundamental groups are zero. The result follows from Lemma 18.16.

The final statement of the theorem follows from Proposition 18.25 below and the obvious fact that a quasirepresentable Deligne-Mumford topological stack is representable. □

Proposition 18.25. *Let $p \colon \mathcal{Y} \to \mathcal{X}$ be a covering map of topological stacks, and assume that $\mathcal{X}$ is a locally connected Deligne-Mumford topological stack. Then, for every point $x \in \mathcal{X}$, there exists an open substack $\mathcal{U} \subseteq \mathcal{X}$ containing x with the following properties:*

i. *$\mathcal{U} \cong [U/I_x]$, for some topological space U acted on by I_x (mildly at x);*
ii. *$p^{-1}(\mathcal{U}) \cong \coprod_{k \in K} [U/G_k]$, where G_k, for k ranging in an index set K, are subgroups of I_x acting on U via I_x.*

In particular, $\mathcal{Y}$ is a Deligne-Mumford topological stack.

Proof. By shrinking $\mathcal{X}$ around x we may assume that $\mathcal{X} = [X/I_x]$, for some locally connected topological space X acted on by I_x (Lemma 18.9). Consider the corresponding étale chart $X \to \mathcal{X}$, and let $Y \to \mathcal{Y}$ be the pull back chart for $\mathcal{Y}$. The map $q \colon Y \to X$, being a pull back of p, is again a covering map. Let x' be the (unique) point in X lying over x. Note that x' is a fixed point of the action of I_x and I_x acts mildly at x'. There is an open set $U \subseteq X$ containing x' over which q trivializes. After replacing U with a smaller open set containing x (say, by the connected component of x in U), we may assume that U is I_x-invariant and connected. Set $\mathcal{U} = [U/I_x]$. We claim that $\mathcal{U}$ has the desired property. Let $\mathcal{V} = p^{-1}(\mathcal{U}) \subseteq \mathcal{Y}$, and $V = q^{-1}(U) \subseteq Y$. Then, V is an étale chart for $\mathcal{V}$ and it is of the form

$$V = \coprod_{j \in J} U_j, \ U_j = U,$$

for some index set J. We can think of V also as an étale chart for $\mathcal{U}$ (by first projecting it down to U via q, and then composing with $U \to \mathcal{U}$). The corresponding groupoid look as follows:

$$\Big[V \times_{\mathcal{U}} V = \coprod_{J\times J\times I_x} U_{j_1,j_2,g} \quad \overset{g\colon U_{j_1,j_2,g}\to U_{j_1}}{\underset{g\colon U_{j_1,j_2,g}\to U_{j_2}}{\rightrightarrows}} \quad \coprod_{j\in J} U_j = V\Big],$$

where $U_{j_1,j_2,g} = U$. We can think of V also as a chart for $\mathcal{V}$. In this case, the corresponding groupoid looks as follows:

$$R := [V \times_{\mathcal{V}} V \rightrightarrows V].$$

Now, the key observation is that, the latter groupoid is an open-closed subgroupoid of the former. This follows by applying Proposition 18.11 to the top left square of the following cartesian diagram:

$$\begin{tikzcd}
V \times_{\mathcal{V}} V \arrow[r] \arrow[d] & V \times_{\mathcal{U}} V \arrow[r] \arrow[d] & V \times V \arrow[d] \\
\mathcal{V} \arrow[r, "\Delta"'] & \mathcal{V} \times_{\mathcal{U}} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{V} \times \mathcal{V} \arrow[d] \\
 & \mathcal{U} \arrow[r] & \mathcal{U} \times \mathcal{U}
\end{tikzcd}$$

Since U is connected, this implies that

$$V \times_{\mathcal{V}} V = \coprod_A U_{j_1,j_2,g},$$

where A is a subset of $J \times J \times I_x$. An immediate conclusion is that, for each slice U_0 in $V = \coprod_{j\in J} U_j$, the orbit $\mathcal{O}(U_0)$ is a disjoint union of copies of some U_j, $j \in J$. By considering orbits of different U_j, $j \in J$, we obtain a partitioning $J = \coprod_{k\in K} J_k$. This gives rise to a decomposition $\mathcal{V} = \coprod_{k\in K} \mathcal{V}_k$. Let us fix a $k \in K$ and see how each individual $\mathcal{V}_k$ looks like. For this, we have to look at the restriction $[R|_{\mathbb{U}_k} \rightrightarrows \mathbb{U}_k]$ of the groupoid $[R \rightrightarrows V]$ to the invariant open $\mathbb{U}_k = \coprod_{j\in J_k} U_j$. Let us fix a representative slice U_{j_k} in $\coprod_{j\in J_k} U_j$. Since all the slices of U in $\coprod_{j\in J_k} U_j$ are permuted around transitively under R, we have $[\mathbb{U}_k/R|_{\mathbb{U}_k}] \cong [U_{j_k}/R|_{U_{j_k}}]$. On the other hand, $R|_{U_{j_k}} = \coprod_{G_k} U_g$, where $G_k \subseteq I_x$ is a subset (we are actually thinking of G_k as the subset $\{j_k\} \times \{j_k\} \times G_k \subseteq J \times J \times I_x$). The fact that $[R|_{U_{j_k}} \rightrightarrows U_{j_k}]$ is a subgroupoid of $[R \rightrightarrows V]$ implies that G_k is in fact a subgroup of I_x, and that the groupoid $[R|_{U_{j_k}} \rightrightarrows U_{j_k}]$ is naturally identified with the action groupoid $[G_k \times U \rightrightarrows U]$ of G_k acting (via I_x) on U. Therefore, we have $\mathcal{V}_k \cong [U_{j_k}/R|_{U_{j_k}}] \cong [U/G_k]$. This completes the proof. $\square$

The example at the end of Section 15 shows that the above proposition fails if we do not assume that $\mathcal{X}$ is locally connected.

19. Examples

In this section we supply some examples of topological stacks. There are five subsections. The first one collects a few pathological examples that can be used here and there as counterexamples. In the next four subsections we consider four general classes of example, namely: gerbes, orbifolds, weighted projective lines, and graphs of groups.

19.1. Some pathological examples.

Example 19.1. Let $X = \mathbb{Z} \cup \{\infty\}$ be the one-point compactification of $\mathbb{Z}$, and let $\mathbb{Z}$ act on X by translation on $\mathbb{Z} \subset X$ and fixing ∞. The corresponding quotient stack $[X/\mathbb{Z}] = \mathcal{X}$ is a connected non locally connected weak Deligne-Mumford topological stack that is not Deligne-Mumford. It has a dense open substack $\mathcal{U}$ that is equivalent to a point. The residue gerbe at ∞ is the closed substack that is complement to $\mathcal{U}$, and it is equivalent to $\mathcal{B}\mathbb{Z}$. The coarse moduli space of $\mathcal{X}$ is a 2 element set $\{0, \infty\}$ with $\{\infty\}$ its only non trivial open set. Note that, although $\mathcal{X}$ is not path connected, it has a path connected coarse moduli space.

Example 19.2. Let c be a non-zero real number. Consider the action of $\mathbb{Z}$ on $\mathbb{R}^2$ given by multiplication by c, and let $\mathcal{X}$ be the quotient stack. Then $\mathcal{X}$ is a path-connected locally path connected weak Deligne-Mumford topological stack that is not Deligne-Mumford. It has an open dense substack $\mathcal{U}$ that is equivalent to a (2-dimensional) torus. The complement of $\mathcal{U}$ is a closed substack equivalent to $\mathcal{B}\mathbb{Z}$ which can be identified as the residue gerbe at 0. The coarse moduli space of $\mathcal{X}$ is homeomorphic to $\mathbb{T}^2 \coprod \{0\}$ where the topology induced on $\mathbb{T}^2$ is the usual topology of a torus, and 0 is in the closure of every point in $\mathbb{T}^2$ (i.e. the only open set containing 0 is the whole space). We have $\pi_1 \mathcal{X} \cong \mathbb{Z}$. All the higher homotopy groups vanish.

Example 19.3. A variation of Example 19.2 is constructed as follows. Consider the vector field $(x, y) \mapsto (x, y)$ on $\mathbb{R}^2$ and look at the corresponding flow. This gives an action of $\mathbb{R}$ on $\mathbb{R}^2$, having 0 as its unique fixed point. The quotient stack $\mathcal{X}$ of this action is a path connected locally path connected topological stack (that is not weak Deligne-Mumford, since the inertia group at 0 is $\mathbb{R}$, which is not discrete). It has an open dense substack $\mathcal{U}$ that is equivalent to the circle S^1. The complement of $\mathcal{U}$ is a closed substack equivalent to $\mathcal{B}\mathbb{R}$ which can be identified as the residue gerbe at 0. The coarse moduli space of $\mathcal{X}$ is homeomorphic to $S^1 \coprod \{0\}$ where the topology induced on S^1 is the usual topology and 0 is in the closure of every point in $\mathbb{T}^2$ (i.e. the only open set containing 0 is the whole space). All homotopy groups of $\mathcal{X}$ vanish.

19.2. Topological gerbes. Any trivial pretopological gerbe is of the form $B_X G$, where X is a topological space and G is a relative topological group over X. This pretopological gerbe is topological (resp., weak Deligne-Mumford, Deligne-Mumford), if the structure map $G \to X$ of the group G is LF (resp., local homeomorphism, resp. covering space).

Recall that any gerbe $\mathcal{X}$ can be covered by open substacks that are trivial gerbes, and $\mathcal{X}$ is topological (resp., weak Deligne-Mumford, Deligne-Mumford), if each of these open substacks are so.

Example 19.4. Let G be a topological group, acting trivially on a topological space X. Then $[X/G] \cong X \times \mathcal{B}G$ is a trivial gerbe (over X). This gerbe is topological for any choice of LF in Example 13.1 except for (**6**). If G is a discrete group, then X/G is Deligne-Mumford.

Example 19.5. Let $\mathcal{X}$ be a topological stack whose coarse moduli space is a single point. Then $\mathcal{X}$ is *not* necessarily a gerbe (Examples 10.4 and 10.5). If $\mathcal{X}$ is Deligne-Mumford, however, then $\mathcal{X}$ is a trivial gerbe (over $*$).

Example 19.6. Let G be a topological group, and $H \subseteq G$ a closed normal subgroup. Let X be a topological space, and Y a G/H-torsor over X. Let G act on Y via G/H. Then $[Y/G]$ is a topological gerbe (over X). This gerbe is trivial if and only Y can be extended to a G-torsor, that is, if there is a G-torsor Z over X such that $Y = Z/H$. In particular, $[Y/G]$ is trivial when the extension

$$1 \to H \to G \to G/H \to 1$$

is split. (Choose a splitting, and define $Z = Y \times_{G/H} G$.)

Example 19.7. In the previous example, take $X = *$ and $Y = G/H$. Then $[Y/G]$ is trivial if and only if the extension

$$1 \to H \to G \to G/H \to 1$$

is split.

Let G be an arbitrary topological group. The source and target maps of the trivial groupoid $[G \rightrightarrows *]$ are **LF** for, say, **LF**=locally cartesian maps (Example 13.1.**3**). So we can talk about homotopy groups of $\mathcal{B}G = [*/G]$. Using the torsor description of $\mathcal{B}G = [*/G]$, it is easy to show that there are natural isomorphisms:

$$\pi_{n-1}(G) \xrightarrow{\sim} \pi_n(\mathcal{B}G), \ n = 1, 2, \cdots .$$

Here, $\pi_0(G)$ stands for the group of *path* components of G (we do not need any assumptions on G); it is isomorphic to G/G_0, where G_0 is the path component of identity. Take $x \colon * \to \mathcal{B}G$ to be the base point of $\mathcal{B}G$. Then, the natural map $\omega_x \colon I_x \to \pi_1(\mathcal{B}G, x)$ is simply the quotient map $G \to G/G_0$.

Using this, we obtain another interpretation of the maps $\omega_x \colon I_x \to \pi_1(\mathcal{X}, x)$, for an arbitrary pointed topological stack $(\mathcal{X}, x)$, as follows.

Let Γ_x be the residue gerbe at x. Then, $\Gamma_x \cong \mathcal{B}\mathbb{I}_x$, where $\mathbb{I}_x$ is naturally a topological group since $\mathcal{X}$ is (pre)topological. The natural pointed map $\Gamma_x \to \mathcal{X}$ induces maps on the homotopy groups

$$\pi_{n-1}(\mathbb{I}_x) = \pi_n(\Gamma_X) \to \pi_n(\mathcal{X}), \ n = 1, 2, \cdots .$$

Setting $n = 1$ and pre-composing with $\mathbb{I}_x \to \mathbb{I}_x/(\mathbb{I}_x)_0$, we obtain a group homomorphism

$$I_x \to \pi_0(G) \to \pi_1(\mathcal{X}).^{10}$$

This homomorphism can be canonically identified with ω_x.

This shows that each homotopy group $\pi_n(\mathcal{X})$ comes with a natural extra structure, namely, the group homomorphism $\pi_{n-1}(\mathbb{I}_x) \to \pi_n(\mathcal{X})$. For this reason, we call $\pi_{n-1}(\mathbb{I}_x)$ the n^{th} *inertial homotopy group*. Inertial homotopy groups appear in the study of the loop stack of a topological stack.

Now, let us compute inertia stacks of topological gerbes. We begin with the following proposition.

[10] Recall that I_x stands for the underlying discrete group of the topological group $\mathbb{I}_x$

Proposition 19.8. *Consider a diagram of discrete groups*

$$\begin{tikzcd} & H \arrow[d, "f"] \\ K \arrow[r, "g"'] & G \end{tikzcd}$$

and set $H' = f(H)$ and $K' = g(K)$. Let A be the the set of double cosets $H'aK'$ (so we have a partitioning $G = \coprod_A H'aK'$). For each double coset $H'aK'$, choose a representative a and let C_a be the subgroup of $H \times K$ defined by

$$C_a = \{(h,k) \mid f(h)\, a\, g(k)^{-1} = a\}.$$

Then, we have a natural 2-cartesian diagram

$$\begin{tikzcd} \coprod_A \mathcal{B}C_a \arrow[r] \arrow[d] & \mathcal{B}H \arrow[d] \\ BK \arrow[r] & \mathcal{B}G \end{tikzcd}$$

This proposition is a special case of the following.

Proposition 19.9. *Consider a diagram of topological groups*

$$\begin{tikzcd} & H \arrow[d, "f"] \\ K \arrow[r, "g"'] & G \end{tikzcd}$$

Consider the action of $H \times K$ on G in which the effect of $(h,k) \in H \times K$ sends $a \in G$ to $f(h)\, a\, g(k)^{-1}$. Then we have a natural 2-cartesian diagram

$$\begin{tikzcd} {[G/(H \times K)]} \arrow[r] \arrow[d] & \mathcal{B}H \arrow[d] \\ BK \arrow[r] & \mathcal{B}G \end{tikzcd}$$

Proof. This follows from the general construction of Section 9. Details left to the reader. □

Corollary 19.10. *Let G be a topological group. Then, the inertia stack of $\mathcal{B}G$ is equivalent to the quotient stack of the conjugation action of G on itself. In particular, when G is discrete, we have*

$$\mathcal{I}\mathcal{B}G \cong \coprod_A \mathcal{B}C_a,$$

where A is the set of conjugacy classes in G, and C_a is the stabilizer group of (a representative a of) a conjugacy class.

Proof. The inertia stack $\mathcal{IB}G$ of $\mathcal{B}G$ is equivalent $\mathcal{B}G \times_{B(G\times G)} \mathcal{B}G$, where the maps $\mathcal{B}G \to B(G \times G)$ are induced from the diagonal map $\Delta\colon G \to G \times G$. So we have an action of $G \times G$ on $X := G \times G$, where an element (g_1, g_2) acts on (x, y) as follows:

$$(x, y) \mapsto (g_1\, x\, g_2^{-1}, g_1\, y\, g_2^{-1}).$$

(We use the notation X for $G \times G$ when it is viewed as a space acted on by $G \times G$.) Hence, $\mathcal{IB}G$ is equivalent to the quotient stack $[X/(G \times G)]$. Consider the subspace $Y = G \times \{1\}$ of X, and let $[R \rightrightarrows Y]$ be the restriction of the action groupoid $[(G \times G) \times X \rightrightarrows X]$. The quotient stack $[Y/R]$ is a substack of $\mathcal{IB}G$. On the other hand, from the fact that $s^{-1}(Y) \to X$ is an epimorphism (in fact, it has a section), it follows that $Y \to \mathcal{IB}G$ is an epimorphism. This implies that $[Y/R] \cong \mathcal{IB}G$ (Proposition 3.4). We leave it to the reader to verify that $[R \rightrightarrows Y]$ is indeed th action groupoid of the conjugation action of G on Y (where Y is identified with G is the obvious way). □

19.3. **Orbifolds.** An *orbifold* in the sense of Thurston [Th] is a Deligne-Mumford topological stack whose isotropy groups are finite and admits an étale chart $p\colon X \to \mathcal{X}$ with X a manifold. It is also assumed that there is an open dense substack of $\mathcal{X}$ that is a manifold.[11] The 2-category **Orb** of orbifolds is a full sub 2-category of the 2-category of Deligne-Mumford topological stacks. It is customary in the literature to identify 2-isomorphic morphisms in **Orb** and work with the resulting 1-category.

The covering theory developed by Thurston in *loc. cit.* is a special case of the covering theory of topological stacks developed in Section 18. This is because of Proposition 18.25. In particular, the existence of a universal cover for an orbifold ([Th], Proposition 13.2.4) follows from Corollary 18.20. Our definition of the fundamental group of a topological stack generalizes Thurston's definition.

What Thurston calls a *good* orbifold is what we have called a *unifromizable* orbifold. We have the following.

Theorem 19.11. *Let $\mathcal{X}$ be a connected orbifold, and let I_x be the inertia group attached to the point $x \in \mathcal{X}$. Then, we have natural group homomorphisms $I_x \to \pi_1(\mathcal{X}, x)$. Furthermore, $\mathcal{X}$ is a good orbifold if and only if for every $x \in \mathcal{X}$ the map $I_x \to \pi_1(\mathcal{X}, x)$ is injective.*

Proof. This follows from Theorem 18.24. □

In [No2] we give a formula for computing the fundamental group of the underlying space (read, coarse moduli space) of a connected orbifold $\mathcal{X}$.[12] Roughly speaking, the formula can be stated as follows. Consider the normal subgroup $N \subseteq \pi_1(\mathcal{X}, x)$ generated by images in $\pi_1(\mathcal{X}, x)$ of $I_{x'}$, for all $x' \in \mathcal{X}$ (for this we have to identify $\pi_1(\mathcal{X}, x')$ with $\pi_1(\mathcal{X}, x)$ by choosing a path from x' to x). Then, $\pi_1(\mathcal{X}, x)/N$ is naturally isomorphic to the fundamental group of the underlying space of $\mathcal{X}$.

[11] Thurston also assumes that X_{mod} is Hausdorff. I am not sure why we need this.

[12] In fact, the formula is valid for a quite general class of topological stacks, including Deligne-Mumford topological stacks. Under some mild conditions, from this we can derive a formula for the fundamental group of the (naive) quotient of a topological group acting on topological space (possibly with fixed points).

19.4. **Weighted projective lines.**

Example 19.12. Let $\mathbb{C}^\times$ act on $\mathbb{C}^2\backslash\{0\}$ by $t\cdot(x,y)=(t^m x, t^n y)$, where m and n are fixed positive integers. The corresponding quotient stack is called the *weighted projective line of weight* (m,n), and is denoted by $\mathcal{P}(m,n)$. It is a Deligne-Mumford topological stack (indeed, it is the topological stack associated to a Deligne-Mumford algebraic stack with the same name). Using the theory of fibrations of topological stacks (which will appear in a sequel to this paper), we can compute homotopy groups of $\mathcal{P}(m,n)$. We have a $\mathbb{C}^\times$-fibration over $\mathcal{P}(m,n)$ whose total space is $\mathbb{C}^2\backslash\{0\}$. A fiber homotopy exact sequence argument shows that $\pi_k\mathcal{P}(m,n)$ is isomorphic to $\pi_k S^2$, for $k\geq 1$. In particular, $\mathcal{P}(m,n)$ is simply connected, for all $m,n\geq 1$. The coarse moduli space of $\mathcal{P}(m,n)$ is homeomorphic to S^2. It has two distinguished points, corresponding to $[1:0]$ and $[0:1]$ in $\mathbb{C}^2$. The residue gerbe at these points are isomorphic to $\mathcal{B}\mathbb{Z}_m$ and $\mathcal{B}\mathbb{Z}_n$, respectively. At every other point the residue gerbe is isomorphic to $\mathcal{B}\mathbb{Z}_d$, where $d=\gcd(m,n)$.

Exercise. Compute the effect of the moduli map $\mathcal{P}(m,n)\to\mathcal{P}(m,n)_{mod}\cong S^2$ on the homotopy groups.

Let us now remove a point (other than the two special points) from $\mathcal{P}(m,n)$. Denote the resulting stack by $\mathcal{U}$. An easy van Kampen argument shows that $\pi_1\mathcal{U}\cong\mathbb{Z}_m *_{\mathbb{Z}_d}\mathbb{Z}_n$. All other homotopy groups of $\mathcal{U}$ vanish. This follows from the fact that, using Theorem 18.24, $\mathcal{U}$ has a universal cover that is a 2-dimensional simply connected manifold. It can be seen that the only possibility is that the universal cover be homeomorphic to $\mathbb{R}^2$.

More details and more examples of this type can be found in [BeNo].

19.5. **Graphs of groups.** In this subsection, we briefly indicate how the theory of *graphs of groups* (for example see [Se], [Ba]) can be embedded in the theory of Deligne-Mumford topological stacks. Many of the basic results of the theory of graphs of groups are more or less immediate consequences of the existence of a homotopy theory for Deligne-Mumford topological stacks.[13]

We start with the simplest case, namely, the graph of groups that looks as follows:

$$G_0 \bullet\!\!\xrightarrow{\quad A\quad}\!\!\bullet\, G_1$$

Here, G_0, G_1 and A are discrete groups, and the incidence of the edge labeled A with the vertex labeled G_i indicates an injective group homomorphism $A\hookrightarrow G_i$.

We would like to think of this as a Deligne-Mumford topological stack $\mathcal{X}$ whose coarse moduli space is the interval $I=[0,1]$. The inertia groups at the end points of this interval are G_0 and G_1, respectively, and the inertia group of any other point in the open interval $(0,1)$ is A. Here is how to make this precise.

Consider the star shaped topological space

$$Y := (G_0/A)\times I\Big/(G_0/A)\times\{0\}.$$

[13]A covering theory for graphs of groups has been developed by Bass in [Ba] in which he points out: *The definition, and verifications of its properties, is regrettably more technical than one might anticipate.* Our theory of covering spaces of topological stacks generalizes Bass's theory.

This is a single point, with a bunch of rays coming out of it, each labeled by a coset of A in G_0. There is a natural action of G_0 on this space, fixing the vertex 0 and permuting the rays emanating from it. This action is mild at 0 (Definition 14.1). Set $\mathcal{Y} = [Y/G_0]$; $\mathcal{Y}$ is a Deligne-Mumford topological stack. Do the similar thing with G_1, namely, take

$$Z := (G_1/A) \times I \Big/ (G_1/A) \times \{1\}$$

and set $\mathcal{Z} = [Z/G_1]$. The complement of the point 0 in $\mathcal{Y}$ is an open substack of $\mathcal{Y}$ that is equivalent to $B_{(0,1)}A \cong BA \times (0,1)$. Similarly, the complement of the point 1 in $\mathcal{Z}$ is an open substack of $\mathcal{Z}$ that is equivalent to $BA \times (0,1)$. We can glue $\mathcal{Y}$ and $\mathcal{Z}$ along these open substacks (Corollary 16.11) to obtain a Deligne-Mumford topological stack $\mathcal{X}$ (Theorem 16.12). This Deligne-Mumford topological stack is what we want to think of as the stacky incarnation of our graphs of groups.

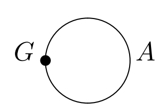
Let us play around a bit with this stack. Using van Kampen theorem (whose proof will appear elsewhere), the fundamental group of $\mathcal{X}$ is easily computed to be isomorphic to $G = G_1 *_A G_2$. Note that, the possible inertial fundamental groups of $\mathcal{X}$ at various points are one of A, G_1 and G_2, and these all inject into G. So by Theorem 18.24, the universal cover of $\mathcal{X}$ is an honest topological space. That is, there is a topological space $X = \tilde{\mathcal{X}}$ with a properly discontinuous action (Definition 14.2) of G such that $[X/G] \cong \mathcal{X}$. By restricting X over the open substacks $\mathcal{Y}$ and $\mathcal{Z}$, we see that X locally looks like either Y or Z; thus, X is a graph. In fact, X is a tree, because it is simply connected. The upshot is that, there is a tree X such that our graph of groups is isomorphic to $[X/G]$, where $G = G_1 *_A G_2$.

This tree is easy to construct explicitly (see [Se]). Take the set of vertices of X to be $V = G/G_1 \coprod G/G_2$, and the set of edges to be $E = G/A$. The end points of edges are determined by maps $G/A \to G/G_0$ and $G/A \to G/G_1$. There is a natural action of G on this graph that has the following properties:

- All the edges are permuted transitively. The vertices are partitioned into two orbits $P_i = G/G_i$, $i = 0, 1$.
- The inertia group of an interior point of an edge is A. The inertia group of a point in P_i is G_i, $i = 1, 2$.

Therefore, the quotient stack $[X/G]$ is equivalent to $\mathcal{X}$.

Another basic graph of groups looks like this:



Here, G and A are discrete groups. The incidence of the two ends of the loop with the vertex correspond to injective group homomorphisms $i, j \colon A \hookrightarrow G$. We can identify A with a subgroup of G via i, and think of j as an injective group homomorphisms $\theta \colon A \to G$.

We will explain shortly how to turn this graph of groups into a Deligne-Mumford topological stack. van Kampen theorem implies that the fundamental group of this graph of groups is isomorphic to the HNN-extension associated to the data (G, A, θ).

In general, a graph of groups is defined to be a graph $\mathcal{G} = (E, V)$, (possibly with multiple edges and loops) equipped with a family G_v of groups, one for each vertex $v \in V$, and a family of groups G_e, one for each edge $e \in E$. We are also given an inclusion $G_v \hookrightarrow G_e$ for an incidence of a vertex v with an edge e.

Using Theorem 16.12, we can turn a graph of groups into a Deligne-Mumford topological stacks. Here is one way of doing it. First, a star-shaped graph of groups (i.e. one having a single vertex in the center with rays coming out of it) can be turned into a Deligne-Mumford topological stack in a way analogous to the very first example of this subsection. For a general graph of groups we do the following. For every edge e, pick two distinct points $e_{\frac{1}{3}}$ and $e_{\frac{2}{3}}$, different from the end points. Let

$$\mathcal{L} = \coprod_{e \in E} \mathcal{B}G_e \times \{\frac{1}{3}, \frac{2}{3}\},$$

$$\mathcal{Y} = \coprod_{e \in E} \mathcal{B}G_e \times [\frac{1}{3}, \frac{2}{3}],$$

$$\mathcal{Z} = \coprod_{v \in V} \mathcal{G}_p,$$

where $\mathcal{G}_p$ is the star-shaped graphs of groups whose center is the vertex p (think of $\mathcal{Z}$ as the graph of groups that remains when we remove the middle third of every edge). There are natural embeddings $\mathcal{L} \hookrightarrow \mathcal{Y}$ and $\mathcal{L} \hookrightarrow \mathcal{Z}$ that can be used to glue $\mathcal{Y}$ and $\mathcal{Z}$ along $\mathcal{L}$ (Theorem 16.12). The glued Deligne-Mumford topological stack is a stacky model for our graph of groups $\mathcal{G}$.

The structure theory for graphs of groups developed by Serre [Se] is best interpreted from the stacky point of view.

The fundamental group of a graph of groups $\mathcal{G}$ ([Se], Section 5.1) is simply the fundamental group of the topological stack associated to $\mathcal{G}$. The existence of the universal cover ([Se], Section 5.3) follows from Corollary 18.20 as follows.

Let $\tilde{\mathcal{G}}$ be the universal cover of $\mathcal{G}$. By Proposition 18.25.**ii**, $\tilde{\mathcal{G}}$ is again a graph of groups. It is easy to check that, for any Deligne-Mumford topological stack $\mathcal{X}$, the induced map $\pi_1(\mathcal{X}) \to \pi_1(X_{mod})$ is surjective (in fact, $\pi_1(X_{mod})$ is explicitly computed in [No2]). This implies that the coarse moduli space of $\tilde{\mathcal{G}}$ is a tree. Now, an easy van Kampen argument shows that, for $\tilde{\mathcal{G}}$ to have trivial fundamental group it is necessary and sufficient that all the groups G_v and G_e be trivial; that is, $\tilde{\mathcal{G}} \cong X$ for some *tree* X. So $\mathcal{G}$ is equivalent to $[X/\pi_1(\mathcal{G})]$, where X is a tree.

The Structure Theorem ([Se], Section 5.4, Theorem 13) is completely obvious from the stacky point of view.

The injectivity of $G_v \to \pi_1(\mathcal{G})$ ([Se], Section 5.2, Corollary 1, whose proof therein is quite tedious and long), is an immediate consequence of Theorem 18.24, because we just showed that the universal cover of $\mathcal{G}$ is an honest topological space (namely, a tree).

The residue gerbes of a graph of groups are easy to figure out: if x is a vertex of $\mathcal{G}$, then the residue gerbe at x is $\mathcal{B}G_x$; if x lies on the interior of an edge e, then the residue gerbe at x is $\mathcal{B}G_e$.

More interesting is the inertia stack $\mathcal{I}\mathcal{G}$ of a graph of groups $\mathcal{G}$. This is again a graph of group whose underlying graph has its set of vertices

$$\{v_{[g]} \mid v \in V,\ [g] \text{ a conj. class in } G_v\}.$$

The set of edges is given by

$$\{e_{[g]} \mid e \in E,\ [g] \text{ a conj. class in } G_e\}.$$

A vertex $v_{[g]}$ is incident with an edge $e_{[h]}$, if v is incident with e, and, under the inclusion $G_v \hookrightarrow G_e$, the conjugacy class $[g]$ maps to the conjugacy class $[h]$.

The group associated to the vertex $v_{[g]}$ is the centralizer $C_g \subseteq G_v$ (we have to make a choice of a representative $g \in [g]$). Similarly, the group associated to the edge $e_{[h]}$ is the centralizer $C_h \subseteq G_e$ (again, we have to make a choice of a representative $h \in [h]$).

For an incident pair $v_{[g]}$ and $e_{[h]}$, the inclusion map $C_h \hookrightarrow C_g$ is defined by the composition

$$C_h \subseteq G_e \hookrightarrow G_v \xrightarrow{\text{conj. by } x} G_v$$

where $x \in G_v$ is an element such that conjugation by x sends (the image in G_v of) h to g; again, we have to make a choice for x. It is easy to see that under this composition C_h lands inside C_g.

Up to equivalence, the resulting stack will be independent of all the choices made.

20. The topological stack associated to an algebraic stack

Take the class **LF** to be any of the Example 13.1 except for (**6**).

All algebraic stacks and schemes considered in this section are assumed to be locally of finite type over $\mathbb{C}$.

In this section we show how to associate a topological stack to an algebraic stack (locally of finite type) over $\mathbb{C}$. That is, we describe how to construct a functor of 2-categories

$$-^{top} \colon \mathsf{AlgSt}_{\mathbb{C}} \to \mathsf{TopSt}.$$

(Of course, this functor factors through the 2-category of analytic stacks, but we will not discuss this here.) We will then prove the stacky version of the *Riemann Existence Theorem*.

Theorem 20.1 ([Mi], Chapter III, Lemma 3.14)**.** *Let X be an algebraic space that is locally of finite type over $\mathbb{C}$, and let X^{top} be the associated topological space. The functor $Y \mapsto Y^{top}$ defines an equivalence between the category of finite étale maps $Y \to X$ and the category of finite covering spaces of X^{top}.*

We remark that the above theorem is stated in [Mi] for passage from schemes to *analytic* spaces. Passage from analytic spaces to topological spaces is straightforward. Also, extending the result from *schemes* to algebraic spaces is easy (and is implicit in what follows).

Sketch of the construction of $\mathcal{X}^{top}$. Assume $\mathcal{X}$ is an algebraic stack that is locally of finite type over $\mathbb{C}$. Let $\mathcal{X} \cong [X/R]$ be a presentation of $\mathcal{X}$ as the quotient of a smooth groupoid $[R \rightrightarrows X]$ (so, X and R are locally of finite type over $\mathbb{C}$). We can now consider the topological groupoid $[R^{top} \rightrightarrows X^{top}]$. Every smooth map of $f \colon X \to Y$ schemes (or algebraic spaces) looks locally (in the étale topology) like a Euclidean projection $\mathbb{A}^m \to \mathbb{A}^n$. That is, for any $x \in X$, after replacing X and Y

by suitable Zariski open sets, the map f fits in a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{étale}} & \mathbb{A}^m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle pr} \\ Y & \xrightarrow[\text{étale}]{} & \mathbb{A}^n \end{array}$$

Here $m \geq n$ and $\mathbb{A}^n$ is identified as the first n coordinates of $\mathbb{A}^m$.

Using the fact that étale maps between schemes induces local homeomorphisms on the associated topological spaces, the above diagram implies that smooth maps induce locally cartesian maps with Euclidean fibers (see Example 13.1.**3′**) on the associated topological spaces. Therefore, the groupoid $[R^{top} \rightrightarrows X^{top}]$ has **LF** source and target maps, where **LF** is any of classes of maps considered in Example 13.1, except for (**6**). We denote the quotient (topological) stack $[X^{top}/R^{top}]$ by $\mathcal{X}^{top}$.

If $X_1 \to \mathcal{X}$ and $X_2 \to \mathcal{X}$ are two different smooth charts for $\mathcal{X}$, with $[R_1 \rightrightarrows X_1]$ and $[R_2 \rightrightarrows X_2]$ the corresponding groupoids, there is always a third one $X_3 \to \mathcal{X}$ dominating both (say $X_3 = X_1 \times_{\mathcal{X}} X_2$, also see Section 8), which gives rise to canonical equivalences $[X_3^{top}/R_3^{top}] \to [X_1^{top}/R_1^{top}]$ and $[X_3^{top}/R_3^{top}] \to [X_2^{top}/R_2^{top}]$. This way we can construct an equivalence $[X_2^{top}/R_2^{top}] \to [X_1^{top}/R_1^{top}]$ which, using a similar argument, can be shown to be independent of the choice of X_3, up to a 2-isomorphism. In other words, $\mathcal{X}^{top}$ is well-defined up to an isomorphism that is unique up to 2-isomorphism.

More generally, given a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ of algebraic stacks, one can realize it as a morphism of groupoids from $[T \rightrightarrows Y]$ to $[R \rightrightarrows X]$. This gives a map $f^{top}\colon \mathcal{Y}^{top} \to \mathcal{X}^{top}$ of topological stacks and again it is easy to show that this map is well-defined up to a unique 2-isomorphism. This, modulo a lot of choices that need to be made (and hence lots of set theoretical problems that are swept under the carpet), will give us the effect of the functor $-^{top}\colon \mathsf{AlgSt}_{\mathbb{C}} \to \mathsf{TopSt}$ on objects and morphisms. Once the choices are made, and the effect of $-^{top}$ on objects and 1-morphisms is determined, the effect of this functor on 2-morphisms can be traced through the construction and be easily seen to be well-defined and uniquely determined. Therefore, we get a functor of 2-categories.

Proposition 20.2. *The functor* $-^{top}\colon \mathsf{AlgSt}_{\mathbb{C}} \to \mathsf{TopSt}$ *commutes with fiber products (more generally, with all finite limits).*

Proof. When all the stacks involved are affine schemes the result is easy to verify because an affine scheme of finite type over $\mathbb{C}$ can be realized as the zero set of a finitely many polynomial in some affine space $\mathbb{A}^n$. The case of fiber products of general schemes can be reduced to the affine case, since both on the algebraic side and the topological side the fiber products can be constructed locally (that is, by constructing fiber products of affine patches and then gluing them together).

The general case of stacks now follows from the construction of fiber products of stacks using their groupoid presentations (Section 9). □

The functor $-^{top}$ has other nice properties: it sends representable morphisms to representable morphisms (use Lemma 6.3), smooth morphisms to local fibrations, étale morphisms to local homeomorphisms, and finite étale morphisms to covering spaces. It also sends Deligne-Mumford stacks (with finite stabilizer) to Deligne-Mumford topological stacks. The latter is due to the fact that a locally Noetherian

Deligne-Mumford stack, with finite stabilizer, has a coarse moduli space and is locally, in the étale topology of its coarse moduli space, a quotient stack of a finite group action.

The following result will be used in the proof of Riemann existence theorem for stacks.

Lemma 20.3. *Let $[R \rightrightarrows Y]$ and $[T \rightrightarrows Y]$ be smooth groupoids (in the category of schemes), and let $f\colon T \to R$ be a map of groupoids (the map $Y \to Y$ being identity) such that the induced map $[Y/T] \to [Y/R]$ on the quotients is (representable) finite étale. Then f is an open-closed embedding. The topological version of the statement is also true.*

Proof. Let $\mathcal{Y} = [Y/T]$ and $\mathcal{X} = [Y/R]$. Then $T = Y \times_{\mathcal{Y}} Y$ and $R = Y \times_{\mathcal{X}} Y$. So we have a 2-cartesian diagram

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d] & R \arrow[d] \\ \mathcal{Y} \arrow[r, "\Delta"'] & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \end{tikzcd}$$

The lower map is an open-closed embedding since $\mathcal{Y} \to \mathcal{X}$ is finite étale, therefore so is f. The topological version is proved similarly (also see Proposition 18.11). $\square$

Now, we are ready to prove the Riemann Existence Theorem for stacks.

Theorem 20.4 (Riemann Existence Theorem for Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack that is locally of finite type over $\mathbb{C}$, and let $\mathcal{X}^{top}$ be the associated topological stack. The functor $\mathcal{Y} \mapsto \mathcal{Y}^{top}$ defines an equivalence between the category of (representable) finite étale maps $\mathcal{Y} \to \mathcal{X}$ and the category of finite covering stacks of $\mathcal{X}^{top}$. (Note that these are honest 1-categories, because we are only considering* representable *maps.)*[14]

Proof. First we prove essential surjectivity. Let $\mathcal{Y}' \to \mathcal{X}^{top}$ a covering map (we use the notation $\mathcal{Y}'$, instead of $\mathcal{Y}$, to emphasis that $\mathcal{Y}'$ is a *topological* stack – in fact, as a matter of notational convention, in this proof every thing that has a superscript is topological, otherwise algebraic). Let $X \to \mathcal{X}$ be a smooth chart for $\mathcal{X}$, with X a scheme. So $X^{top} \to \mathcal{X}^{top}$ will be an LF chart for $\mathcal{X}^{top}$. Let $Y' = \mathcal{Y}' \times_{\mathcal{X}^{top}} X^{top}$ be the pull-back chart for $\mathcal{Y}'$. We can think of the composition $Y' \to X^{top} \to \mathcal{X}^{top}$ as an LF chart for $\mathcal{X}^{top}$. Set $T' = Y' \times_{\mathcal{Y}'} Y'$ and $R' = Y' \times_{\mathcal{X}^{top}} Y'$. By (the topological version of) Lemma 20.3, T' is an open-closed subspace of R'. Since $Y' \to X^{top}$ is a covering space, by the classical version of the Riemann Existence Theorem, there exists a scheme Y, locally of finite type over $\mathbb{C}$, such that $Y^{top} = Y'$. Let $R = Y \times_{\mathcal{X}} Y$. By Proposition 20.2, we have $R^{top} \cong R'$. It is obvious that the decomposition of R into its connected components is preserved under $-^{top}$. Therefore, $T' = T^{top}$ for some open-closed subspace of R. It is easy to see that $[T \rightrightarrows Y]$ is a subgroupoid of $[R \rightrightarrows Y]$. If we let $\mathcal{Y} = [Y/T]$, then $\mathcal{Y} \to \mathcal{X}$ is finite étale, and $\mathcal{Y}^{top} \to \mathcal{X}^{top}$ is isomorphic, as a covering space of $\mathcal{X}^{top}$, to $\mathcal{Y}' \to \mathcal{X}^{top}$. This proves the essential surjectivity.

[14] More precisely, 2-categories in which there is at most one 2-isomorphism between any two 1-morphisms.

Next we prove fully-faithfulness. Let $\mathcal{Y}$ and $\mathcal{Z}$ be finite étale covers of $\mathcal{X}$. We may assume $\mathcal{Y}$ is connected. To give a map from $\mathcal{Y}$ to $\mathcal{Z}$ relative to $\mathcal{X}$ is the same is to specify a connected component of $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ that maps isomorphically to $\mathcal{Y}$. Using Proposition 20.2, this is the same as to specify a connected component of $\mathcal{Y}^{top} \times_{\mathcal{X}^{top}} \mathcal{Z}^{top}$ that maps isomorphically to $\mathcal{Y}^{top}$, and these are of course in bijection with maps from $\mathcal{Y}^{top}$ to $\mathcal{Z}^{top}$ relative to $\mathcal{X}^{top}$. This proves the fully-faithfulness. $\square$

Corollary 20.5. *Let $\mathcal{X}$ be a connected algebraic stack that is locally of finite type over $\mathbb{C}$. Assume $\mathcal{X}^{top}$ is locally path connected and semilocally 1-connected. Then we have an isomorphism*

$$\pi_1^{alg}(\mathcal{X}) \to \widehat{\pi_1(\mathcal{X}^{top})}.$$

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