GROUP COHOMOLOGY WITH COEFFICIENTS IN A CROSSED-MODULE

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Abstract. We compare three different ways of defining group cohomology with coefficients in a crossed-module: 1) explicit approach via cocycles; 2) geometric approach via gerbes; 3) group theoretic approach via butterflies. We discuss the case where the crossed-module is braided and the case where the braiding is symmetric. We prove the functoriality of the cohomologies with respect to weak morphisms of crossed-modules and also prove the "long" exact cohomology sequence associated to a short exact sequence of crossed-modules and weak morphisms.

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1. Introduction

These notes grew out as an attempt to answer certain questions regarding group cohomology with coefficients in a crossed-module which were posed to us by M. Borovoi in relation to his work on abelian Galois cohomology of reductive groups [Bor].

We collects some known or not so-well-known results in this area and put them in a coherent (and hopefully user-friendly) form, as well as add our own new approach to the subject via butterflies. We hope that the application-minded user finds these notes beneficial. We especially expect these result to be useful in Galois cohomology (e.g., in the study of relative Picard groups of Brauer groups).

Let us outline the content of the paper. Let Γ be a group acting strictly on a crossed-module $\mathbb{G}$. We investigate the group cohomologies $H^i(\Gamma, \mathbb{G})$. We compare three different ways of constructing the cohomologies $H^i(\Gamma, \mathbb{G})$, $i = -1, 0, 1$. One approach is entirely new. We also work out some novel aspects of the other two approaches which, to our knowledge, were not considered previously.

Let us briefly describe the three approaches that we are considering.

The cocycle approach. The first approach uses an explicit cocycle description of the cohomology groups. Many people have worked on this. The original idea goes back to Dedecker [Ded1, Ded2]. But he only considers the case of a trivial Γ-action (where things get oversimplified). Borovoi [Bor] treats the general case in his study of abelianization of Galois cohomology of reductive groups. A systematic approach is developed in [CeFe] where the more general case of a 2-group fibered over a category is treated. The cohomology groups with coefficients in a symmetric braided crossed-module have been studied in [CaMa, BuCaCe, Ulb]. The paper of Garzón and del Río [GadR] seems to be the first place where the group structure on H^1 appears in print. Breen's letter to Borovoi [Bre4] also discusses the group structure on H^1 in the crossed-module language and gives explicit formulas. Also relevant is the work [BGPT] in which the authors study homotopy types of equivariant crossed-complexes.

We also point out that there is a standard way of going from Čech cohomology to group cohomology, as discussed in ([Bre3], § 5.7). In this way, it is possible in principle to deduce results about group cohomology from Breen's general results on Čech cohomology.

In Sections 3-4 we rework the definitions of H^i, $i = -1, 0, 1$. The only originality we may claim in these sections is merely in the form of presentation (e.g., explicitly working out all the formulas in the language of crossed-modules), as the concepts are well understood.

What seems to be original here is that in §5 we introduce an explicit crossed-module in groupoids concentrated in degrees $[-1, 1]$, denoted $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$, whose cohomologies are precisely $H^i(\Gamma, \mathbb{G})$. This crossed-module in groupoids encodes everything that is known about the H^i (and more).

In the case where $\mathbb{G}$ is endowed with a Γ-equivariant braiding, we show that $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$ is a 2-crossed-module. In particular, $H^1(\Gamma, \mathbb{G})$ is a group and $H^i(\Gamma, \mathbb{G})$ are abelian, $i = -1, 0$. When the braiding is symmetric, we show that $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$ is a braided 2-crossed-module. This implies that $H^1(\Gamma, \mathbb{G})$ is also abelian.

We also prove that $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$ is functorial in (strict) Γ-equivariant morphisms of crossed-modules and takes an equivalence of crossed-modules to an equivalence of crossed-modules in groupoids (or of 2-crossed-modules, resp., braided 2-crossed-modules, in the case where $\mathbb{G}$ is braided, resp., symmetric). In particular, an equivalence of crossed-modules induces an isomorphism on all H^i.

The butterfly approach. In the second approach, we construct a 2-groupoid $\mathfrak{Z}(\Gamma, \mathbb{G})$ such that $H^i(\Gamma, \mathbb{G}) \cong \pi_{1-i}\mathfrak{Z}(\Gamma, \mathbb{G})$. The objects of this 2-groupoid are certain diagrams of groups involving Γ and the G_i; see §7.1. In §7.2 we give an sketch of how to construct a biequivalence between $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$ and the crossed-module in groupoids associated to $\mathfrak{Z}(\Gamma, \mathbb{G})$.

We show that $\mathfrak{Z}(\Gamma, \mathbb{G})$ is functorial in weak Γ-equivariant morphisms $\mathbb{H} \to \mathbb{G}$ of crossed-modules (read *strong* Γ*-equivariant butterflies*). In particular, it takes an equivalence of butterflies to an equivalence of 2-groupoids. In the case where $\mathbb{G}$ is endowed with a Γ-equivariant braiding, we endow $\mathfrak{Z}(\Gamma, \mathbb{G})$ with a natural monoidal structure which makes it a group object in the category of 2-groupoids and weak functors. Under the equivalence between $\mathfrak{Z}(\Gamma, \mathbb{G})$ and $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$, the group structure on the former corresponds to the 2-crossed-module structure on the latter. In the case where the braiding on $\mathbb{G}$ is symmetric, $\mathfrak{Z}(\Gamma, \mathbb{G})$ admits a symmetric braiding.

The gerbe approach. Finally, the third approach is that of Breen [Bre1] adopted to our specific situation (it is also closely related to [CeFe]). We construct another 2-groupoid $\mathfrak{Z}(\Gamma, \mathcal{G})$ which we show is naturally biequivalent to $\mathfrak{Z}(\Gamma, \mathbb{G})$. Here, $\mathcal{G}$ is the 2-group associated to $\mathbb{G}$. The objects of this 2-groupoid are principal $\mathcal{G}$-bundles over the classifying stack $B\Gamma$ of Γ. We show that $\mathfrak{Z}(\Gamma, \mathcal{G})$ is functorial in weak Γ-equivariant morphisms $\mathbb{H} \to \mathbb{G}$ of crossed-modules (read Γ-equivariant butterflies).

In the case where $\mathbb{G}$ is endowed with a Γ-equivariant braiding, $\mathfrak{Z}(\Gamma, \mathcal{G})$ admits a natural monoidal structure which makes it a group object in the category of 2-groupoids and weak functors. In this case, the equivalence between $\mathfrak{Z}(\Gamma, \mathcal{G})$ and $\mathfrak{Z}(\Gamma, \mathbb{G})$ is monoidal. When the braiding on $\mathbb{G}$ is symmetric, $\mathfrak{Z}(\Gamma, \mathcal{G})$ admits a symmetric braiding and so does the equivalence between $\mathfrak{Z}(\Gamma, \mathcal{G})$ and $\mathfrak{Z}(\Gamma, \mathbb{G})$.

We also consider the last two approaches in the case where everything is over a Grothendieck site. This is useful for geometric applications in which Γ and $\mathbb{G}$ are topological, Lie, algebraic, and so on.

Finally, we show that a short exact sequence

$$1 \to \mathbb{K} \to \mathbb{H} \to \mathbb{G} \to 1$$

of Γ-crossed-modules and weakly Γ-equivariant weak morphisms (read Γ-butterflies) over a Grothendieck site gives rise to a long exact cohomology sequence

$$1 \to H^{-1}(\Gamma,\mathbb{K}) \to H^{-1}(\Gamma,\mathbb{H}) \to H^{-1}(\Gamma,\mathbb{G}) \to H^{0}(\Gamma,\mathbb{K}) \to$$
$$H^{0}(\Gamma,\mathbb{H}) \longrightarrow H^{0}(\Gamma,\mathbb{G}) \longrightarrow H^{1}(\Gamma,\mathbb{K}) \longrightarrow H^{1}(\Gamma,\mathbb{H}) \to H^{1}(\Gamma,\mathbb{G}).$$

Also see [CeGa] and [CeFe], Theorem 31.

One last comment. We end this introduction by pointing out one serious omission in this paper: H^2. In the case where $\mathbb{G}$ has a Γ-equivariant braiding, one expects to be able to push the theory one step further to include H^2. In the first approach, the complex $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ is expected to be the ≤ 1 truncation of a certain complex $\mathcal{K}^{\leq 2}(\Gamma,\mathbb{G})$ concentrated in degrees $[-1,2]$. In the second and the third approaches, the 2-groupoids $\mathfrak{Z}(\Gamma,\mathbb{G})$ and $\mathfrak{Z}(\Gamma,\mathcal{G})$ get replaced by certain pointed 3-groupoids whose automorphism 2-groupoids of the base object are $\mathfrak{Z}(\Gamma,\mathbb{G})$ and $\mathfrak{Z}(\Gamma,\mathcal{G})$. The long exact cohomology sequence should also extend to include the three additional H^2 terms. The machinery for doing all this is being developed in a forthcoming paper [AlNo3] and is not available yet in print. For that reason, we will not get into the discussion of H^2 in these notes.

All of the above can also be done with the group Γ replaced by a crossed-module (the action on $\mathbb{G}$ remains strict). This is useful because in some applications (e.g., in Galois cohomology) the action of Γ on $\mathbb{G}$ is not strict but it can be replaced with a strict action of a 2-group equivalent to Γ. We will not pursue this topic here and leave it to a future paper.

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2. Notation and conventions

Let $\mathbb{G} = [\partial\colon G_1 \to G_0]$ be a crossed-module. We assume that G_0 acts on G_1 on the right. We denote the action of $g \in G_0$ on $\alpha \in G_1$ by α^g. Let Γ be a group acting on a crossed-module $\mathbb{G} = [\partial\colon G_1 \to G_0]$ on the left. We denote the action of $\sigma \in \Gamma$ on an element g by ${}^{\sigma}g$. We require the Γ action on $\mathbb{G}$ to be compatible with the action of G_0 on G_1 in the following way:

$${}^{\sigma}(\alpha^g) = ({}^{\sigma}\alpha)^{{}^{\sigma}g}.$$

We usually denote $({}^{\sigma}\alpha)^g$ by ${}^{\sigma}\alpha^g$. Note that this is *not* equal to ${}^{\sigma}(\alpha^g)$.

Our convention for braiding $\{,\}\colon G_0 \times G_0 \to G_1$ is that $\partial\{g,h\} = g^{-1}h^{-1}gh$. The braidings are assume to be Γ-equivariant in the sense that $\{{}^{\sigma}g, {}^{\sigma}h\} = {}^{\sigma}\{g,h\}$, for every $g,h \in G_0$ and $\sigma \in \Gamma$.

Whenever there is fear of confusion, we use a dot $\cdot$ for products in complicated formulas; the same products may appear without a dot in other places (even in the same formulas).

All groupoids, 2-groupoids and so on are assumed to be small.

3. H^{-1} and H^0 of a Γ-equivariant crossed-module

By definition, $H^{-1}(\Gamma, \mathbb{G}) = (\ker \partial)^{\Gamma}$. This is an abelian group. Let us now define $H^0(\Gamma, \mathbb{G})$.

A 0**-cochain** is a pair (g, θ) where $g \in G_0$ and $\theta \colon \Gamma \to G_1$ is a pointed map. We denote the set of 0-cochain by $C^0(\Gamma, \mathbb{G})$. There is a multiplication on $C^0(\Gamma, \mathbb{G})$ which makes it into a group. By definition, the product of two 0-cochains (g_1, θ_1) and (g_2, θ_2) is

$$(g_1, \theta_1)(g_2, \theta_2) := (g_1 g_2, \theta_1^{g_2} \theta_2),$$

where $\theta_1^{g_2}\theta_2 \colon \Gamma \to G_1$ is defined by $\sigma \mapsto \theta_1(\sigma)^{g_2}\theta_2(\sigma)$.

Remark 3.1. In the case where $\mathbb{G}$ is braided, there is another way of making $C^0(\Gamma, \mathbb{G})$ into a group. This will be discussed in §3.1 and used later on in §4.

A 0-cochain (g, θ) is a 0**-cocycle** if the following conditions are satisfied:

- For every $\sigma \in \Gamma$, $\partial\theta(\sigma) = g^{-1} \cdot {}^{\sigma}g$,
- For every $\sigma, \tau \in \Gamma$, $\theta(\sigma\tau) = \theta(\sigma) \cdot {}^{\sigma}\theta(\tau)$.

The 0-cocycles forms a subgroup of $C^0(\Gamma, \mathbb{G})$ which we denote by $Z^0(\Gamma, \mathbb{G})$.

An element in $Z^0(\Gamma, \mathbb{G})$ is a 0**-coboundary** if it is of the form $(\partial\mu, \theta_\mu)$, where $\mu \in G_1$ and $\theta_\mu \colon \Gamma \to G_1$ is defined by $\theta_\mu(\sigma) := \mu^{-1} \cdot {}^{\sigma}\mu$. It is easy to see that the set $B^0(\Gamma, \mathbb{G})$ of 0-coboundaries is a normal subgroup of $Z^0(\Gamma, \mathbb{G})$; it is in fact normal in $C^0(\Gamma, \mathbb{G})$ too. We define

$$H^0(\Gamma, \mathbb{G}) := \frac{Z^0(\Gamma, \mathbb{G})}{B^0(\Gamma, \mathbb{G})}.$$

This group is not in general abelian.

A better way of phrasing the above discussion is to say that

$$[G_1 \to Z^0(\Gamma, \mathbb{G})]$$

$$\mu \mapsto (\partial\mu, \theta_\mu)$$

is a crossed-module. The action of $Z^0(\Gamma, \mathbb{G})$ on G_1 is defined by

$$\mu^{(g,\theta)} := \mu^g.$$

3.1. In the presence of a braiding on $\mathbb{G}$.

When $\mathbb{G}$ is braided, $H^0(\Gamma, \mathbb{G})$ is abelian. This is true thanks to the following.

Lemma 3.2. *The commutator of the two* 0*-cocycles* (g, θ) *and* (g', θ') *in* $Z^0(\Gamma, \mathbb{G})$ *is equal to the* 0*-coboundary* $(\partial\mu, \theta_\mu)$*, where* $\mu = \{g, g'\}$.

In fact, it follows from the above lemma that the bracket

$$\{(g, \theta), (g', \theta')\} := \{g, g'\}$$

makes the crossed-module $[G_1 \to Z^0(\Gamma, \mathbb{G})]$ defined at the end of the previous subsection into a braided crossed-module.

As we pointed out in Remark 3.1, in the presence of a braiding on $\mathbb{G}$, there is a second product on $C^0(\Gamma, \mathbb{G})$ which makes it into a group as well. This new product will be used in an essential way in §4. Here is how it is defined. Given two

0-cochains (g_1,θ_1) and (g_2,θ_2), their product is the 0-cochain (g_1g_2,ϑ), where ϑ is defined by the formula

$$\vartheta(\sigma) := \theta_1(\sigma)^{p_2(\sigma)}\cdot\theta_2(\sigma)\cdot\{g_2^{-1},g_1^{-1}{}^{\sigma}g_1\}^{{}^{\sigma}g_2}$$

Here, $p_2(\sigma)=g_2^{-1}\cdot{}^{\sigma}g_2\cdot\partial\theta_2(\sigma)^{-1}$.

It is not hard to check that, when restricted to $Z^0(\Gamma,\mathbb{G})$, the above product coincides with the one defined in the previous subsection.

4. H^1 of a Γ-equivariant crossed-module

A 1-cocycle on Γ with values in $\mathbb{G}$ is a pair (p,ε) where

$$p\colon\Gamma\to G_0 \quad\text{and}\quad \varepsilon\colon\Gamma\times\Gamma\to G_1$$

are pointed set maps satisfying the following conditions:

- For every $\sigma,\tau\in\Gamma$, $p(\sigma\tau)\cdot\partial\varepsilon(\sigma,\tau)=p(\sigma)\cdot{}^{\sigma}p(\tau)$.
- For every $\sigma,\tau,\upsilon\in\Gamma$, $\varepsilon(\sigma,\tau\upsilon)\cdot{}^{\sigma}\varepsilon(\tau,\upsilon)=\varepsilon(\sigma\tau,\upsilon)\cdot\varepsilon(\sigma,\tau)^{{}^{\sigma\tau}p(\upsilon)}$.

We denote the set of 1-cocycles by $Z^1(\Gamma,\mathbb{G})$. This is a pointed set with the base point being the pair of constant functions $(1_{G_0},1_{G_1})$. In fact, $Z^1(\Gamma,\mathbb{G})$ is the set of objects a groupoid $\mathcal{Z}^1(\Gamma,\mathbb{G})$. An arrow

$$(p_1,\varepsilon_1)\to(p_2,\varepsilon_2)$$

in $\mathcal{Z}^1(\Gamma,\mathbb{G})$ is given by a pair (g,θ), with $g\in G_0$ and $\theta\colon\Gamma\to G_1$ a pointed map, such that

- for every $\sigma\in\Gamma$,
$$p_2(\sigma)=g^{-1}\cdot p_1(\sigma)\cdot{}^{\sigma}g\cdot\partial\theta(\sigma)^{-1};$$
- for every $\sigma,\tau\in\Gamma$,
$$\varepsilon_2(\sigma,\tau)=\theta(\sigma\tau)\cdot\varepsilon_1(\sigma,\tau)^{{}^{\sigma\tau}g}\cdot\big(\theta(\sigma)^{-1}\big)^{({}^{\sigma}g^{-1}.{}^{\sigma\tau}g)}\cdot{}^{\sigma}\theta(\tau)^{-1}.$$

The above formulas can be interpreted as a right action of the group $C^0(\Gamma,\mathbb{G})$ (with the group structure introduced at the beginning of §3) on the set $Z^1(\Gamma,\mathbb{G})$. We denote this right action by

$$(p_2,\varepsilon_2)=(p_1,\varepsilon_1)^{(g,\theta)}.$$

The groupoid $\mathcal{Z}^1(\Gamma,\mathbb{G})$ is simply the transformation groupoid of this action.

We define $H^1(\Gamma,\mathbb{G})$ to be the pointed set of isomorphism classes of the groupoid $\mathcal{Z}^1(\Gamma,\mathbb{G})$.

4.1. **In the presence of a braiding on $\mathbb{G}$.** In the previous subsection, we constructed the groupoid $\mathcal{Z}^1(\Gamma,\mathbb{G})$ of 1-cocycles as the transformation groupoid of a certain action of $C^0(\Gamma,\mathbb{G})$ on $Z^1(\Gamma,\mathbb{G})$. In the case where $\mathbb{G}$ is endowed with a Γ-equivariant braiding, we will see below that the set $Z^1(\Gamma,\mathbb{G})$ itself also has a group structure. In this situation, it is natural to ask is whether the action of $C^0(\Gamma,\mathbb{G})$ on $Z^1(\Gamma,\mathbb{G})$ is a (right) multiplication action via a certain group homomorphism $C^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G})$.

The answer to this question appears to be negative. However, if we use the alternative group structure on $C^0(\Gamma,\mathbb{G})$ that we introduced in §3.1, then there does exist such a group homomorphism $d\colon C^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G})$. We point out that the right multiplication action of $C^0(\Gamma,\mathbb{G})$ on $Z^1(\Gamma,\mathbb{G})$ obtained via d is *different* from the action discussed in the previous subsection. But, fortunately, the resulting transformation groupoids are the same (Lemma 4.1). In particular, $H^1(\Gamma,\mathbb{G})$ is equal to the cokernel of d.

The product in $Z^1(\Gamma,\mathbb{G})$. We define a product in $Z^1(\Gamma,\mathbb{G})$ as follows. More generally, let $C^1(\Gamma,\mathbb{G})$ be the set of all 1-cochains, where by a **1-cochain** we mean a pair (p,ε),

$$p\colon \Gamma \to G_0, \quad \varepsilon\colon \Gamma\times\Gamma \to G_1,$$

of pointed set maps. Let (p_1,ε_1) and (p_2,ε_2) be in $C^1(\Gamma,\mathbb{G})$. We define the product $(p_1,\varepsilon_1)\cdot(p_2,\varepsilon_2)$ to be the pair (p,ε) where

$$p(\sigma) := p_1(\sigma)p_2(\sigma),$$

$$\varepsilon(\sigma,\tau) := \varepsilon_1(\sigma,\tau)^{p_2(\sigma\tau)}\cdot\varepsilon_2(\sigma,\tau)\cdot\{p_2(\sigma),{}^{\sigma}p_1(\tau)\}^{{}^{\sigma}p_2(\tau)}.$$

It can be checked that this makes $C^1(\Gamma,\mathbb{G})$ into a group. The inverse of the element (p,ε) in $C^1(\Gamma,\mathbb{G})$ is the pair (q,λ) where

$$q(\sigma) := p(\sigma)^{-1},$$

$$\lambda(\sigma,\tau) = (\varepsilon(\sigma,\tau)^{-1})^{p(\sigma\tau)^{-1}}\cdot\{p(\sigma)^{-1},{}^{\sigma}p(\tau)^{-1}\}.$$

The subset $Z^1(\Gamma,\mathbb{G})\subset C^1(\Gamma,\mathbb{G})$ is indeed a subgroup.

The group homomorphism $d\colon C^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G})$. Next we construct a group homomorphism

$$d\colon C^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G}).$$

(Note that $C^0(\Gamma,\mathbb{G})$ is endowed with the group structure defined in §3.1.) Let (g,θ) be in $C^0(\Gamma,\mathbb{G})$. We define $d(g,\theta)$ to be the pair (p,ε) where

$$p(\sigma) := g^{-1}\cdot{}^{\sigma}g\cdot\partial\theta(\sigma)^{-1},$$

$$\varepsilon(\sigma,\tau) := \theta(\sigma\tau)\cdot\big(\theta(\sigma)^{-1}\big)^{({}^{\sigma}g^{-1}\cdot{}^{\sigma\tau}g)}\cdot{}^{\sigma}\theta(\tau)^{-1}$$

It is not difficult to check that this is a group homomorphism.

The crossed-module $[d\colon C^0/B^0\to Z^1]$. The group homomorphism d vanishes on the subgroup $B^0(\Gamma,\mathbb{G})\subseteq C^0(\Gamma,\mathbb{G})$ of 0-coboundaries. Therefore, d factors through a homomorphism

$$d\colon C^0(\Gamma,\mathbb{G})/B^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G}),$$

which, by abuse of notation, we have denoted again by d. There is a right action of $Z^1(\Gamma,\mathbb{G})$ on $C^0(\Gamma,\mathbb{G})$ which preserves $B^0(\Gamma,\mathbb{G})$ and makes

$$[d\colon C^0(\Gamma,\mathbb{G})/B^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G})]$$

into a crossed-module. It is given by

$$(g,\theta)^{(p,\varepsilon)} = (g,\vartheta),$$

where $\vartheta\colon\Gamma\to G_1$ is defined by

$$\vartheta(\sigma) = \theta(\sigma)^{p(\sigma)}\cdot\{p(\sigma),{}^{\sigma}g\}\cdot\{g,p(\sigma)\}^{g^{-1}\cdot{}^{\sigma}g}.$$

Observe that the kernel of d coincides with $H^0(\Gamma,\mathbb{G})$. We show that the cokernel of d coincides with $H^1(\Gamma,\mathbb{G})$. We do so by comparing the action of $C^0(\Gamma,\mathbb{G})$ on $Z^1(\Gamma,\mathbb{G})$ introduced in the previous subsection (the one that gave rise to the groupoid $\mathfrak{Z}^1(\Gamma,\mathbb{G})$ of 1-cocycles) with the multiplication action of $C^0(\Gamma,\mathbb{G})$ on $Z^1(\Gamma,\mathbb{G})$ via d. More precisely, we have the following.

Lemma 4.1. *Let (g,θ) be in $C^0(\Gamma,\mathbb{G})$ and (p,ε) in $Z^1(\Gamma,\mathbb{G})$. Let $(p,\varepsilon)^{(g,\theta)}$ be the action of $C^0(\Gamma,\mathbb{G})$ on $Z^1(\Gamma,\mathbb{G})$ introduced at the end of the previous subsection. Then,*

$$(p,\varepsilon)^{(g,\theta)} = (p,\varepsilon)\cdot d\big(g,\theta\cdot\delta(p,g)\big),$$

where $\delta(p,g)\colon \Gamma\to G_1$ is defined by

$$\sigma\mapsto \{g,p(\sigma)\}^{g^{-1}\cdot{}^{\sigma}g}.$$

Corollary 4.2. *When $\mathbb{G}$ has a Γ-equivariant braiding, the first cohomology set $H^1(\Gamma,\mathbb{G})$ inherits a natural group structure, $H^0(\Gamma,\mathbb{G})$ is abelian, and there is a natural action of $H^1(\Gamma,\mathbb{G})$ on $H^0(\Gamma,\mathbb{G})$.*

Remark 4.3. The crossed-module $[d\colon C^0/B^0\to Z^1]$ is a model for the 2-group $\mathcal{H}^1$ defined by Garzón and del Río [GadR].

4.2. **When braiding is symmetric.** In the previous subsection, we saw that when $\mathbb{G}$ has a Γ-equivariant braiding, the first cohomology set $H^1(\Gamma,\mathbb{G})$ carries a natural group structure. This was done by identifying $H^1(\Gamma,\mathbb{G})$ with the cokernel of the crossed-module $[d\colon C^0/B^0\to Z^1]$. In the case where the braiding is symmetric (i.e., $\{g,h\}\{h,g\}=1$) we can do even better.

Lemma 4.4. *Suppose that the braiding on $\mathbb{G}$ is symmetric (resp., Picard). Then, the crossed-module $[d\colon C^0(\Gamma,\mathbb{G})/B^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G})]$ is braided and symmetric (resp., Picard). The braiding is given by*

$$\{(p_1,\varepsilon_1),(p_2,\varepsilon_2)\} := (1,\{p_2,p_1\}),$$

where $\{p_2,p_1\}\colon\Gamma\to G_1$ is the pointwise bracket of the maps $p_1,p_2\colon\Gamma\to G_0$. (Note the reverse order.)

The above braiding is obtained by unraveling the symmetry morphism b of §7.5.

Corollary 4.5. *When the braiding on $\mathbb{G}$ is symmetric, the group structure on $H^1(\Gamma,\mathbb{G})$ is abelian.*

5. The nonabelian complex $\mathcal{K}(\Gamma,\mathbb{G})$

For an abelian group G with an action of Γ one can find a chain complex $\mathcal{K}(\Gamma,G)$ whose cohomologies are $H^i(\Gamma,G)$. The corresponding statement is obviously not true for a nonabelian G (or a crossed-module $\mathbb{G}$). However, it seems to be true "as much as it can be". More precisely, even though the complex $\mathcal{K}(\Gamma,\mathbb{G})$ does not exist, truncated versions of it exist. And "the more abelian $\mathbb{G}$ is" the longer and the more abelian these truncations become.

For example, for an arbitrary $\mathbb{G}$, there is a crossed-module in groupoids $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$, concentrated in degrees $[-1,1]$, whose cohomologies are precisely $H^i(\Gamma,\mathbb{G})$, $i=-1,0,1$. When $\mathbb{G}$ is braided, $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ is actually a 2-crossed-module; in fact, it can be extended one step further to a 2-crossed-module in groupoids $\mathcal{K}^{\leq 2}(\Gamma,\mathbb{G})$ which is concentrated in degrees $[-1,2]$.[1] In the case where $\mathbb{G}$ is symmetric, $\mathcal{K}^{\leq 2}(\Gamma,\mathbb{G})$ is expected to come from a 3-crossed-module. We are not able to prove this here, but we prove the weaker statement that $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ is a braided 2-crossed-module.

[1] We will not prove this here.

5.1. **The case of arbitrary $\mathbb{G}$.** The crossed-module in groupoids $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ that we will define below neatly packages everything we have discussed so far in the preceding sections.

Let $\mathcal{Z}(\Gamma,\mathbb{G}) = [Z^1(\Gamma,\mathbb{G}) \times C^0(\Gamma,\mathbb{G}) \rightrightarrows Z^1(\Gamma,\mathbb{G})]$ be the groupoid of 1-cocycles defined in §4. Recall that it is the action groupoid of the right action of C^0 on Z^1 defined in §4. We define $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ to be

$$\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G}) := [\coprod_{c\in Z^1} G_1(c) \xrightarrow{d} \mathcal{Z}(\Gamma,\mathbb{G})].$$

Here $Z^1 = Z^1(\Gamma,\mathbb{G})$ and $G_1(c) = G_1$.

For every 1-cocycle $c \in Z^1(\Gamma,\mathbb{G})$, the effect of the differential d on the corresponding component $G_1(c)$ of the disjoint union $\coprod_{c\in Z^1} G_1(c)$ is defined by

$$\mu \mapsto (\partial_\mu, \theta_\mu).$$

(See §3 for notation.) Here, we are thinking of $(\partial_\mu,\theta_\mu)$ as an arrow in $\mathcal{Z}(\Gamma,\mathbb{G})$ going from the object c to itself.

The right action of $\mathcal{Z}(\Gamma,\mathbb{G})$ on $\coprod_{c\in Z^1} G_1(c)$ is defined as follows. Let $c, c' \in Z^1$ be 1-cocycles, and $(g,\theta) \in C^0$ an arrow between them. Then, (g,θ) acts by

$$G_1(c) \to G_1(c'),$$
$$\mu \mapsto \mu^{(g,\theta)} := \mu^g.$$

The following proposition is easy to prove.

Proposition 5.1. *For $i = -1, 0, 1$, we have*

$$H^i(\Gamma,\mathbb{G}) = H^i\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G}).$$

5.2. **The case of braided $\mathbb{G}$.** In the case where $\mathbb{G}$ is endowed with a Γ-equivariant braiding, it follows from Lemma 4.1 that the crossed-module in groupoids $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ comes from (see §12) the 2-crossed-module

$$[C^{-1}(\Gamma,\mathbb{G}) \xrightarrow{d} C^0(\Gamma,\mathbb{G}) \xrightarrow{d} Z^1(\Gamma,\mathbb{G})],$$

where $C^{-1}(\Gamma,\mathbb{G}) := G_1$. (Note that the group structure on $C^0(\Gamma,\mathbb{G})$ is the one defined in §3.1.) The boundary maps d are the ones defined in §3 and §4. That is

$$C^{-1}(\Gamma,\mathbb{G}) \xrightarrow{d} C^0(\Gamma,\mathbb{G}),$$
$$\mu \mapsto (\partial_\mu, \theta_\mu),$$

and

$$C^0(\Gamma,\mathbb{G}) \xrightarrow{d} Z^1(\Gamma,\mathbb{G}),$$
$$(g,\theta) \mapsto (p,\varepsilon),$$

where

$$p(\sigma) := g^{-1}\cdot {}^{\sigma}g \cdot \partial\theta(\sigma)^{-1},$$
$$\varepsilon(\sigma,\tau) := \theta(\sigma\tau)\cdot \left(\theta(\sigma)^{-1}\right)^{({}^{\sigma}g^{-1}.{}^{\sigma\tau}g)} \cdot {}^{\sigma}\theta(\tau)^{-1}.$$

The action of $Z^1(\Gamma,\mathbb{G})$ on $C^{-1}(\Gamma,\mathbb{G})$ is defined to be the trivial one. The action of $Z^1(\Gamma,\mathbb{G})$ on $C^0(\Gamma,\mathbb{G})$ is defined to be the one of § 4.1. Namely,

$$(g,\theta)^{(p,\varepsilon)} := (g,\vartheta),$$

where $\vartheta\colon \Gamma \to G_1$ is defined by

$$\vartheta(\sigma) = \theta(\sigma)^{p(\sigma)} \cdot \{p(\sigma), {}^{\sigma}g\} \cdot \{g, p(\sigma)\}^{g^{-1}.{}^{\sigma}g}.$$

The action of $C^0(\Gamma,\mathbb{G})$ on $C^{-1}(\Gamma,\mathbb{G})$ is defined by

$$\mu^{(g,\theta)} := \mu^g.$$

Finally, the bracket

$$\{,\}\colon C^0(\Gamma,\mathbb{G})\times C^0(\Gamma,\mathbb{G})\to C^{-1}(\Gamma,\mathbb{G})$$

is defined by

$$\{(g_1,\theta_1),(g_2,\theta_2)\} := \{g_1,g_2\}.$$

By abuse of notation, we denote the above 2-crossed-module again by $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$. By Proposition 5.1, the cohomologies of this 2-crossed-module are naturally isomorphic to $H^i(\Gamma,\mathbb{G})$, $i=-1,0,1$.

5.3. **The case of symmetric $\mathbb{G}$.** In the case where the braiding on $\mathbb{G}$ is symmetric, the 2-crossed-module $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ is braided in the sense of §12. We prove this using the following lemma.

Lemma 5.2. *Let $\mathbb{C}=[K\xrightarrow{\partial}L\xrightarrow{\partial}M]$ be a 2-crossed-module such that the action of M on K is trivial and the bracket $\{,\}\colon L\times L\to K$ is symmetric (i.e., $\{g,h\}\{h,g\}=1$, for every $g,h\in L$). Assume that we are given a bracket $\{,\}\colon M\times M\to L$ which satisfies the following conditions:*

- *for every $x,y\in M$, $\partial\{x,y\}=x^{-1}y^{-1}xy$,*
- *for every $g\in L$ and $x\in M$, $\{\partial g,x\}=g^{-1}g^x$ and $\{x,\partial g\}=(g^{-1})^x g$.*

With the notation of §12, let the brackets $\{,\}_{(1,0)(2)}$, $\{,\}_{(2,0)(1)}$, $\{,\}_{(0)(2,1)}$, $\{,\}_{(0)(2)}$ be the trivial ones (i.e., their value is always 1). Let $\{,\}_{(2)(1)}\colon L\times L\to K$ be the given bracket of $\mathbb{C}$, and define $\{,\}_{(1)(0)}\colon L\times L\to K$ by $\{g,h\}_{(1)(0)} := \{h,g\}_{(2)(1)}$. Then, $\mathbb{C}$ is a braided 2-crossed-module in the sense of §12.

Proof. All axioms (**3CM1**)-(**3CM18**) of ([ArKuUs], Definition 8) follow trivially from our assumptions, except for **3CM6**. For this, we must prove that

$$\{g,h\}^{[g,h]}=\{g,h\}$$

for every $g,h\in L$. Here, $[g,h]:=g^{-1}h^{-1}gh$. We have, $[g,h]=\partial\{g,h\}(h^{-1})^{\partial g}h$. Note that the assertion is true if we replace $[g,h]$ by $\partial\{g,h\}$. So we have to show that $\{g,h\}^{(h^{-1})^{\partial g}h}=\{g,h\}$. This is true because the action of M on K is trivial and $\partial\colon K\to L$ is M-equivariant. □

Remark 5.3. Note that we are using modified versions of axioms of ([ArKuUs], Definition 8) because our conventions for the actions (left or right) and the brackets, hence also our 2-crossed-module axioms, are different from those of *loc. cit.*

Now, in the above lemma take $\mathbb{C}$ to be $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$. Let

$$\{,\}\colon Z^1(\Gamma,\mathbb{G})\times Z^1(\Gamma,\mathbb{G})\to C^0(\Gamma,\mathbb{G}).$$

be the braiding defined in Lemma 4.4. Namely,

$$\{(p_1,\varepsilon_1),(p_2,\varepsilon_2)\} := (1,\{p_2,p_1\}).$$

Here, $\{p_1,p_2\}\colon\Gamma\to G_1$ is the pointwise bracket of the maps $p_1,p_2\colon\Gamma\to G_0$. It is not difficult to check that this bracket satisfies the two conditions of the above lemma. This endows $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ with the structure of a braided 2-crossed-module.

5.4. **Invariance under equivalence.** The crossed-module in groupoids $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$ is clearly functorial in (strict) Γ-equivariant morphisms $f\colon \mathbb{H} \to \mathbb{G}$ of crossed-modules. (Also, in the braided case, the 2-crossed-module $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$ is functorial in strict Γ-equivariant braided morphisms of crossed-modules.) In this subsection, we prove that if f is an equivalence of crossed-modules (i.e., induces isomorphisms on cohomologies), then the induced morphism

$$f_*\colon \mathcal{K}^{\leq 1}(\Gamma, \mathbb{H}) \to \mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$$

is also an equivalence (i.e., induces isomorphisms on cohomologies).

We begin by a definition. Let

$$\mathbb{G} = [\partial\colon G_1 \to G_0]$$

be a crossed-module, and $p\colon G_0' \to G_0$ a group homomorphism from a certain group G_0'. Let $G_1' := G_1 \times_{G_0} G_0'$ be the fiber product. There is a natural crossed-module structure on

$$\mathbb{G}' := [\mathrm{pr}_2\colon G_1' \to G_0'].$$

We call this the *pullback* crossed-module via p and denote it by $p^*\mathbb{G}$. We have a natural projection $P\colon \mathbb{G}' \to \mathbb{G}$.

The next lemmas will be used in the proof of Proposition 5.6.

Lemma 5.4. *Notation being as above, assume that the images of p and ∂ generate G_0. Then $P\colon \mathbb{G}' \to \mathbb{G}$ is an equivalence of crossed-modules. Conversely, if $P\colon \mathbb{G}' \to \mathbb{G}$ is an equivalence of crossed-modules, then the images of $P_0\colon G_0' \to G_0$ and ∂ generate G_0, and $\mathbb{G}'$ is naturally isomorphic to $P_0^*\mathbb{G}$.*

Proof. Easy. □

Lemma 5.5. *Let $F\colon \mathbb{H} \to \mathbb{G}$ be an equivalence of Γ-crossed-modules. Then, there is a commutative diagram*

$$\mathbb{H} \xleftarrow{P} \mathbb{H}' \xrightarrow{F'} \mathbb{G}, \qquad F\colon \mathbb{H} \to \mathbb{G}$$

of equivalences of Γ-crossed-modules such that P_0 and F_0' are surjective. In particular, by Lemma 5.4, $\mathbb{H}'$ is naturally isomorphic to both $P_0^*\mathbb{H}$ and $(F_0')^*\mathbb{G}$.

Proof. Consider the right action of H_0 on G_1 via $F_0\colon H_0 \to G_0$, and form the semidirect product $H_0 \ltimes G_1$. It acts on $H_1 \times G_1$ on the right by the rule

$$(\beta, \alpha)^{(h,\gamma)} := (\beta^h, \gamma^{-1}\alpha^{P_0(h)}\gamma).$$

With this action, we obtain a Γ-crossed-module

$$\mathbb{H}' := [\partial\colon H_1 \times G_1 \to H_0 \ltimes G_1],$$

$$\partial(\beta, \alpha) := (\partial_{\mathbb{H}}\beta, F_1(\beta^{-1})\alpha).$$

We define $P\colon \mathbb{H}' \to \mathbb{H}$ to be the first projection map $(\mathrm{pr}_1, \mathrm{pr}_1)$, and $F'\colon \mathbb{H}' \to \mathbb{G}$ to be (pr_2, ρ), where $\rho\colon H_0 \ltimes G_1 \to G_0$ is defined by $(h, \alpha) \mapsto P_0(h)\partial_{\mathbb{G}}\alpha$. It is easy to verify that P and F' satisfy the desired properties. □

We now come to the proof of invariance of $\mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$ under equivalences.

Proposition 5.6. *Let $f\colon \mathbb{H}\to\mathbb{G}$ be a Γ-equivariant morphism of crossed-modules which is an equivalence (i.e., induces isomorphisms on* $\ker\partial$ *and* $\operatorname{coker}\partial$*). Then,*

$$f_*\colon \mathcal{K}^{\leq 1}(\Gamma,\mathbb{H})\to\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$$

is an equivalence of crossed-modules in groupoids (i.e., induces isomorphisms on cohomologies). In particular, the induced maps $H^i(\Gamma,\mathbb{H})\to H^i(\Gamma,\mathbb{G})$ are isomorphisms for $i=-1,0,1$.

Proof. By Lemma 5.5, we may assume that $f_0\colon H_0\to G_0$ is surjective. Therefore, by Lemma 5.4, we may assume that $\mathbb{H}$ is the pullback of $\mathbb{G}$ along f_0. That is, $\mathbb{H}=[\mathrm{pr}_1\colon H_0\times_{G_0}G_1\to H_0]$ and f is $(\mathrm{pr}_2,f_0)\colon [H_0\times_{G_0}G_1\to H_0]\to[G_1\to G_0]$. We calculate $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{H})$ explicitly and show that f_* induces isomorphisms on cohomologies.

By definition (§5.1), we have

$$\mathcal{K}^{\leq 1}(\Gamma,\mathbb{H})=\coprod_{c\in Z^1}H_0\times_{G_0}G_1(c)\stackrel{d}{\longrightarrow}\mathcal{Z}(\Gamma,\mathbb{H}),$$

where

$$\mathcal{Z}(\Gamma,\mathbb{H})=[Z^1(\Gamma,\mathbb{H})\times C^0(\Gamma,\mathbb{H})\rightrightarrows Z^1(\Gamma,\mathbb{H})].$$

We calculate $Z^1(\Gamma,\mathbb{H})$ and $C^0(\Gamma,\mathbb{H})$ as follows. An element in $Z^1(\Gamma,\mathbb{H})$ is a pair (p,ε) where

$$p\colon\Gamma\to H_0 \quad\text{and}\quad \varepsilon\colon\Gamma\times\Gamma\to G_1$$

are pointed set maps satisfying the following conditions:

- for every $\sigma,\tau\in\Gamma$, $f_0p(\sigma\tau)\cdot\partial\varepsilon(\sigma,\tau)=f_0p(\sigma)\cdot{}^{\sigma}f_0p(\tau)$,
- for every $\sigma,\tau,\upsilon\in\Gamma$, $\varepsilon(\sigma,\tau\upsilon)\cdot{}^{\sigma}\varepsilon(\tau,\upsilon)=\varepsilon(\sigma\tau,\upsilon)\cdot\varepsilon(\sigma,\tau)^{{}^{\sigma\tau}p(\upsilon)}$.

An element in $C^0(\Gamma,\mathbb{H})$ is a triple $\big(h,(\theta_1,\theta_2)\big)$, with $h\in H_0$, $\theta_1\colon\Gamma\to H_0$, and $\theta_2\colon\Gamma\to G_1$ pointed set maps such that $f_0\theta_1=\partial\theta_2$.

The map of groupoids $\mathcal{Z}(\Gamma,\mathbb{H})\to\mathcal{Z}(\Gamma,\mathbb{G})$ is the one induced by the following maps:

$$Z^1(\Gamma,\mathbb{H})\to Z^1(\Gamma,\mathbb{G}),$$
$$(p,\varepsilon)\mapsto(f_0p,\varepsilon),$$
$$C^0(\Gamma,\mathbb{H})\to C^0(\Gamma,\mathbb{G}),$$
$$\big(h,(\theta_1,\theta_2)\big)\mapsto\big(f_0(h),\theta_2\big).$$

Since f_0 is surjective, we see immediately that $\mathcal{Z}(\Gamma,\mathbb{H})\to\mathcal{Z}(\Gamma,\mathbb{G})$ is surjective on objects and that it is a fibration of groupoids (i.e., has the arrow lifting property). This almost proves that the induced map on the set of isomorphism classes of these groupoids is a bijection (which is the same thing as saying that f_* induces a bijection on the first cohomology sets). All we need to check is that if $(p,\varepsilon),(p',\varepsilon')\in Z^1(\Gamma,\mathbb{H})$ map to the same element in $Z^1(\Gamma,\mathbb{G})$, then they are joined by an arrow in $\mathcal{Z}(\Gamma,\mathbb{H})$. We have $\varepsilon=\varepsilon'$ and $f_0p=f_0p'$. Hence, the map $\theta\colon\Gamma\to H_0$ defined by $\sigma\mapsto p(\sigma)^{-1}p'(\sigma)$ factors through $\ker f_0$. It is easy to see that $\big(1,(1,\theta)\big)\in C^0(\Gamma,\mathbb{H})$ provides the desired arrow in $\mathcal{Z}(\Gamma,\mathbb{H})$ joining (p,ε) to (p',ε'). This completes the proof that f_* is a bijection on the first cohomology sets.

To show that f_* induces an isomorphism on H^{-1} and H^0, we need to verify that the induced map of crossed-modules (see §3)

$$[H_0\times_{G_0}G_1\to Z^0(\Gamma,\mathbb{H})]\ \to\ [G_1\to Z^0(\Gamma,\mathbb{G})]$$

is an equivalence. Observe that $Z^0(\Gamma,\mathbb{H}) \subset C^0(\Gamma,\mathbb{H})$ consists of triples $\big(h,(\theta_h,\theta_2)\big)$, where h and θ_2 are arbitrary and $\theta_h\colon \Gamma \to H_0$ is defined by the rule $\sigma \mapsto h^{-1}\cdot {}^{\sigma}h$. The map $Z^0(\Gamma,\mathbb{H}) \to Z^0(\Gamma,\mathbb{G})$ is given by $\big(h,(\theta_h,\theta_2)\big) \mapsto \big(p_0(h),\theta_2\big)$. It is clear that this map is surjective.

By Lemma 5.4, it is enough to show that the following diagram is cartesian

$$\begin{array}{ccc} H_0 \times_{G_0} G_1 & \xrightarrow{d'} & Z^0(\Gamma,\mathbb{H}) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow \\ G_1 & \xrightarrow[d]{} & Z^0(\Gamma,\mathbb{G}) \end{array}$$

The fact that this diagram is cartesian becomes obvious once we recall (§3) that d and d' are defined as follows:

$$d(\mu) = (\partial\mu,\theta_\mu),\ \ d'(h,\alpha) = \big(h,(\theta_h,\theta_\alpha)\big).$$

The proof of the proposition is complete. □

Remark 5.7. The butterfly approach of §6 provides another proof of Proposition 5.6.

6. Butterflies

This section is a prelude to §7 in which we will present an alternative construction of the cohomologies $H^i(\Gamma,\mathbb{G})$ and also of the complex $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$. This new construction, which is based on the idea of *butterfly*, has the following advantages: 1) it is much easier to write down $H^i(\Gamma,\mathbb{G})$ and describe their properties, 2) it is easy to recover the cocycles from this description, 3) it works for arbitrary topological groups, and in fact in any topos, 4) it can be generalized to the case where Γ itself is a crossed-module.

To motivate the relevance of butterflies, let us explain the idea in the case of H^1. Assume for the moment that $\mathbb{G} = G$ is a group. In this case, to give a 1-cocycle (e.g, a crossed-homomorphism) $p\colon \Gamma \to G$ is the same thing as giving a group homomorphism $\tilde{p}\colon \Gamma \to G \rtimes \Gamma$ making the following diagram commutative:

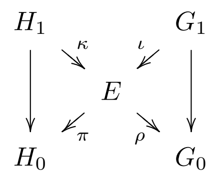
$$\begin{array}{ccc} & & G \rtimes \Gamma \\ & {\scriptstyle \tilde{p}}\nearrow & \downarrow{\scriptstyle \mathrm{pr}} \\ \Gamma & \xrightarrow[\mathrm{id}]{} & \Gamma \end{array}$$

The (right) conjugation action of $G \subseteq G \rtimes \Gamma$ on $G \rtimes \Gamma$ induces an action on the set of such $\tilde{p}$. The transformation groupoid of this action is what we called $\mathcal{Z}^1(\Gamma,G)$ in §4. The set of isomorphism classes of $\mathcal{Z}^1(\Gamma,G)$ is $H^1(\Gamma,G)$.

The aim is now to imitate this definition in the case where G replaced by a crossed-module $\mathbb{G}$. A group homomorphisms $\tilde{p}\colon \Gamma \to G \rtimes \Gamma$ should now be replaced by a *weak* morphism of crossed-modules $\Gamma \to \mathbb{G} \rtimes \Gamma$. This is where butterflies come in the picture.

6.1. **Butterflies.** We recall the definition of a butterfly from [Noo2].

Let $\mathbb{G} = [G_2 \to G_1]$ and $\mathbb{H} = [H_2 \to H_1]$ be crossed-modules. By a *butterfly* from $\mathbb{H}$ to $\mathbb{G}$ we mean a commutative diagram of groups



such that the two diagonal maps are complexes and the NE-SW diagonal is short exact. We require that for every $x \in E$, $\alpha \in G$ and $\beta \in H$,

$$\iota(\alpha^{\rho(x)}) = x^{-1}\iota(\alpha)x \text{ and } \kappa(\beta^{\pi(x)}) = x^{-1}\kappa(\beta)x.$$

We denote the above butterfly by the 5-tuple $(E, \rho, \pi, \iota, \kappa)$, or if there is no fear of confusion, simply by E.

A *morphism* between two butterflies $(E, \rho, \pi, \iota, \kappa)$ and $(E', \rho', \pi', \iota', \kappa')$ is a pair (t, g) where $g \in G_0$ and $t \colon E \to E'$ is an isomorphism of groups. We require that t commutes with the κ and the π maps and satisfies the relations

$$g^{-1}\rho(x)g = \rho'(t(x)) \text{ and } \iota'(\alpha^g) = t\iota(\alpha)$$

for every $x \in E$, $\alpha \in G_1$. The composition of two arrows $(t, g) \colon E \to E'$ and $(t', g') \colon E' \to E''$ is defined to be $(t' \circ t, gg')$.

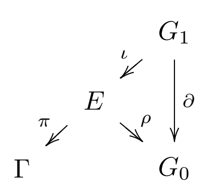
A *2-morphism* between (g, t) and (g', t') is an element $\mu \in G_1$ such that

$$g\partial(\mu) = g' \text{ and } t' = \mu^{-1}t\mu.$$

The composition of two 2-arrows μ_1 and μ_2 is defined to be $\mu_1\mu_2$.

For fixed $\mathbb{H}$ and $\mathbb{G}$, the butterflies between them are objects of a 2-groupoid whose morphisms and 2-arrows are defined as above.

Example 6.1. Assume that $\mathbb{H} = \Gamma$ is a group. Then, a butterfly from Γ to $\mathbb{G}$ is a diagram



where the diagonal sequence is short exact and the map ρ intertwines the conjugation action of E on G_1 with the crossed-module action of G_0. Such a diagram corresponds to a weak morphism $\Gamma \to \mathbb{G}$ and also to a 1-cocycle on Γ with values in $\mathbb{G}$ (for the trivial action of Γ). A morphism between two such diagrams corresponds to a transformation of weak functors and also to an equivalence of 1-cocycles.

Example 6.2. A braided crossed-module $\mathbb{G}$ is the same things as a group object in the category of crossed-modules and weak morphisms. More precisely, the multiplication morphism of this group object is given by the butterfly

$$\begin{array}{ccccc} G_1 \times G_1 & & & & G_1 \\ & \overset{k}{\searrow} & & \overset{i}{\swarrow} & \\ {\scriptstyle(\partial,\partial)}\downarrow & & B & & \downarrow{\scriptstyle\partial} \\ & \underset{p}{\swarrow} & & \underset{r}{\searrow} & \\ G_0 \times G_0 & & & & G_0 \end{array}$$

where the group B is defined as follows. The underlying set of B is $G_0 \times G_0 \times G_1$. The product in B is defined by

$$(g,h,\alpha)\cdot(g',h',\alpha') := (gg', hh', \{h,g'\}^{h'}\alpha^{g'h'}\alpha').$$

The structure maps of the butterfly are given by

$$k(\alpha,\beta) := (\partial\alpha, \partial\beta, \beta^{-1}\alpha^{-1}), \quad i(\alpha) := (1,1,\alpha)$$
$$p(g,h,\alpha) := (g,h), \quad r(g,h,\alpha) := gh\partial\alpha,$$

7. Cohomology via butterflies

In this section we will use the idea discussed at the beginning of the previous section to give a simple description of the cohomologies $H^i(\Gamma,\mathbb{G})$.

Given a group Γ acting on a crossed-module $\mathbb{G}$ on the left, the semi-direct product $\mathbb{G} \rtimes \Gamma$ is the crossed-module

$$\mathbb{G} \rtimes \Gamma := [(\partial,1)\colon G_1 \to G_0 \rtimes \Gamma].$$

The action of $G_0 \rtimes \Gamma$ on G_1 is defined by

$$\alpha^{(g,\sigma)} := {}^{\sigma^{-1}}(\alpha^g) = ({}^{\sigma^{-1}}\alpha)^{({}^{\sigma^{-1}}g)}.$$

This crossed-module comes with a natural projection map

$$\mathrm{pr}\colon \mathbb{G} \rtimes \Gamma \to \Gamma.$$

Here, by abuse of notation, we have denoted the crossed-module $[1 \to \Gamma]$ by Γ.

7.1. **Butterfly description of $H^i(\Gamma,\mathbb{G})$.** We define the 2-groupoid $\mathfrak{Z}(\Gamma,\mathbb{G})$ as follows. The set of objects of $\mathfrak{Z}(\Gamma,\mathbb{G})$ are pairs (E,ρ) where E is an extension

$$1 \to G_1 \xrightarrow{\iota} E \xrightarrow{\pi} \Gamma \to 1$$

and $\rho\colon E \to G_0$ is a map which makes the diagram

$$\begin{array}{ccccc} & & & & G_1 \\ & & & \overset{\iota}{\swarrow} & \\ & & E & & \downarrow{\scriptstyle\partial} \\ & \underset{\pi}{\swarrow} & & \underset{\rho}{\searrow} & \\ \Gamma & & & & G_0 \end{array}$$

commute and satisfies the following conditions:

- for every $x,y \in E$, $\rho(xy) = \rho(x)\cdot {}^{\pi(x)}\rho(y)$,
- for every $x \in E$ and $\alpha \in G_1$, $\iota\big({}^{\pi(x)^{-1}}(\alpha^{\rho(x)})\big) = x^{-1}\iota(\alpha)x$.

(The maps ι and π are also part of the data but we suppress them from the notation. We usually identify G_0 with $\iota(G_0) \subseteq E$ and denote $\iota(\mu)$ simply by μ.)

An arrow in $\mathfrak{Z}(\Gamma, \mathbb{G})$ from (E, ρ) to (E', ρ') is a pair (t, g) where $g \in G_0$ and t is an isomorphism $t\colon E \to E'$ such that

- $\pi = \pi' \circ t$,
- For every $x \in E$, $g^{-1} \cdot \rho(x) \cdot {}^{\pi(x)}g = \rho' t(x)$,
- For every $\alpha \in G_1$, $\iota'(\alpha^g) = t\iota(\alpha)$.

The composition of two arrows $(t, g)\colon (E, \rho) \to (E', \rho')$ and $(t', g')\colon (E', \rho') \to (E'', \rho'')$ is defined to be $(t' \circ t, gg')$.

A 2-arrow $(t, g) \Rightarrow (t', g')$ is an element $\mu \in G_1$ such that $g\partial(\mu) = g'$ and $t' = \mu^{-1}t\mu$. The composition of the two 2-arrows $\mu\colon (t, g) \Rightarrow (t', g')$ and $\mu'\colon (t', g') \Rightarrow (t'', g'')$ is defined to be $\mu\mu'$.

The 2-groupoid is naturally pointed. The base object is (E_{triv}, ρ_{triv}), where $E_{triv} = G_1 \rtimes \Gamma$ and $\rho_{triv}\colon E_{triv} \to G_0$ sends (α, σ) to $\partial(\alpha)$.

Let us now explain how $H^i(\Gamma, \mathbb{G})$, $i = -1, 0, 1$, can be recovered from the 2-groupoid $\mathfrak{Z}(\Gamma, \mathbb{G})$.

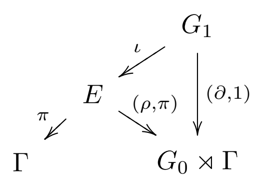
The group $H^{-1}(\Gamma, \mathbb{G})$ is naturally isomorphic to the group of 2-arrows from the arrow $(\mathrm{id}_{E_{triv}}, 1_{G_0})$ to itself.

The group $H^0(\Gamma, \mathbb{G})$ is naturally isomorphic to the group of 2-isomorphism classes of arrows from the base object (E_{triv}, ρ_{triv}) to itself.

The pointed set $H^1(\Gamma, \mathbb{G})$ is naturally isomorphic to the pointed set of isomorphism classes of objects in $\mathfrak{Z}(\Gamma, \mathbb{G})$.

We can also describe the groupoid $\mathcal{Z}^1(\Gamma, \mathbb{G})$ defined in §4. This groupoid is naturally equivalent to the groupoid obtained by identifying 2-isomorphic arrows in $\mathfrak{Z}(\Gamma, \mathbb{G})$.

Remark 7.1. By associating the one-winged butterfly



to an object in $\mathfrak{Z}(\Gamma, \mathbb{G})$, the 2-groupoid $\mathfrak{Z}(\Gamma, \mathbb{G})$ defined above is seen to be isomorphic to the 2-groupoid of butterflies from Γ to $\mathbb{G} \rtimes \Gamma$ whose composition with the projection map $\mathbb{G} \rtimes \Gamma \to \Gamma$ is equal to the identity map $\Gamma \to \Gamma$. Note that, in contrast with [Noo2], here we are considering *non pointed* transformation between butterflies. That is why we obtain a 2-groupoid (rather than a groupoid) of butterflies.

7.2. **Relation to the cocycle description of H^i.** To see how to recover a cocycle in the sense of §4 from the pair (E, ρ), *choose* a set theoretic section $s\colon \Gamma \to E$ to the map π. Assume $s(1) = 1$. Define $p\colon \Gamma \to G_0$ to be the composition $\rho \circ s$ and $\varepsilon\colon \Gamma \times \Gamma \to G_1$ to be

$$\varepsilon\colon (\sigma, \tau) \;\mapsto\; s(\sigma\tau)^{-1}s(\sigma)s(\tau).$$

The pair (p, ε) is a 1-cocycle in the sense of §4. Conversely, given a 1-cocycle (p, ε) in the sense of §4, we define E to be the group that has $\Gamma \times G_1$ as the underlying set and whose product is defined by

$$(\sigma_1,\alpha_1)\cdot(\sigma_2,\alpha_2) := \big(\sigma_1\sigma_2\,,\ \varepsilon(\sigma_1,\sigma_2)\cdot {}^{\sigma_2^{-1}}(\alpha_1^{p(\sigma_2)})\alpha_2\big).$$

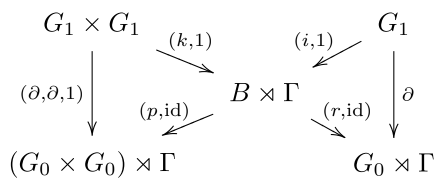
Define the group homomorphism $\rho\colon E\to G_0$ by

$$\rho(\sigma,a)=p(\sigma)\partial({}^{\sigma}g).$$

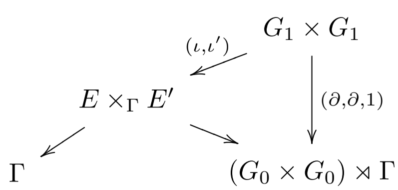
The homomorphisms $\iota\colon G_1\to E$ and $\pi\colon E\to\Gamma$ are the inclusion and the projection maps on the corresponding components.

7.3. **Group structure on $\mathfrak{Z}(\Gamma,\mathbb{G})$; the preliminary version.** In the case where $\mathbb{G}$ has a Γ-equivariant braiding, there is a group structure on the 2-groupoid $\mathfrak{Z}(\Gamma,\mathbb{G})$ which lifts the one on $H^1(\Gamma,\mathbb{G})$ introduced in §4.1. In this subsection, we illustrate this product by making use of the butterfly of Example 6.2. In §7.4, we give an explicit formula for it.

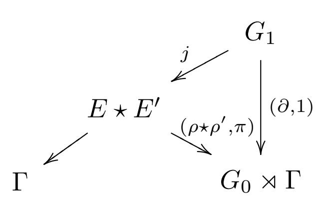
We begin by observing that the butterfly of Example 6.2 can be augmented to a butterfly



This butterfly gives rise to a product on $\mathfrak{Z}(\Gamma,\mathbb{G})$ as follows. Given (E,ρ) and (E',ρ'), we can think of them as one-winged butterflies from Γ to $\mathbb{G}\rtimes\Gamma$ relative to Γ (see Remark 7.1). Form the one-winged diagonal butterfly from Γ to the fiber product of $\mathbb{G}\rtimes\Gamma$ with itself relative to Γ. That is, consider



where $E\times_\Gamma E'$ stands for the fiber product of E and E' over Γ. Composing this butterfly with the one of the beginning of this subsection, we find a one-winged butterfly



This is the sought after product $(E,\rho)\star(E',\rho')$. (In §7.4 we will explicitly write down what $(E,\rho)\star(E',\rho')$ is.)

The above product makes $\mathfrak{Z}(\Gamma,\mathbb{G})$ into a (weak) group object in the category of 2-groupoids and weak functors. Therefore, $\mathfrak{Z}(\Gamma,\mathbb{G})$ corresponds to a (weak) 3-group. Homotopy theoretically, a 3-group is equivalent to a 2-crossed-module. The 2-crossed-module $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ that we encountered in §5.2 is a model for the 3-group

$\mathfrak{Z}(\Gamma,\mathbb{G})$. More precisely, the construction introduced at the end of §7.1 gives an equivalence from the 3-group associated to $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ to $\mathfrak{Z}(\Gamma,\mathbb{G})$.

If we identify 2-isomorphic arrows in $\mathfrak{Z}(\Gamma,\mathbb{G})$, we obtain a (weak) group object in the category of groupoids, i.e., a (weak) 2-group. This 2-group is, in turn, equivalent to the 2-group associated to the crossed-module

$$[d\colon C^0(\Gamma,\mathbb{G})/B^0(\Gamma,\mathbb{G}) \to Z^1(\Gamma,\mathbb{G})],$$

namely to $\mathcal{Z}(\Gamma,\mathbb{G})$ (§4). The set $H^1(\Gamma,\mathbb{G})$ of isomorphism classes of $\mathfrak{Z}(\Gamma,\mathbb{G})$ also inherits a group structure. This group structure coincides with the one defined in §4.1.

7.4. **Group structure on $\mathfrak{Z}(\Gamma,\mathbb{G})$; the explicit version.** In this subsection, we explicitly write down the multiplication in $\mathfrak{Z}(\Gamma,\mathbb{G})$. The formulas are obtained by unraveling the definition of the composition of butterflies ([Noo2], §10.1). It is more or less straightforward how to derive the formulas, but if done naively one usually ends up with very involved expressions. Some extra algebraic manipulation is needed to bring the formulas to the form presented below.

Let us start with the product of two objects in $\mathfrak{Z}(\Gamma,\mathbb{G})$. The product $(E,\rho)\star(E',\rho')$ is the pair $(E\star E',\rho\star\rho')$ which is defined as follows. Let $F := E\times_\Gamma E'$ be the fiber product of E and E' over Γ. We endow F with the following group structure:

$$(x_1,y_1)\cdot(x_2,y_2) := \big(x_1x_2\cdot\iota\{\rho'(y_1)^{-1},{}^{\pi(x_1)}\rho(x_2)\}^{-1}, y_1y_2\big).$$

There is a normal subgroup of F consisting of elements of the form $(\iota(\alpha),\iota'(\alpha)^{-1})$, $\alpha\in G_1$. We define $E\star E'$ to be the quotient of F by this normal subgroup. Alternatively, one can think of $E\star E'$ as the group obtained from F by declaring $(\iota(\alpha),1)$ equal to $(1,\iota'(\alpha))$, for every $\alpha\in G_1$. There is a natural group homomorphism $j\colon G_1\to E\star E'$ which sends α to the common value of $(\iota(\alpha),1)$ and $(1,\iota'(\alpha))$. The group homomorphism $\rho\star\rho'\colon E\star E'\to G_0$ is defined by

$$\rho\star\rho'\colon (x,y)\mapsto \rho(x)\rho'(y).$$

This completes the definition of the product $(E,\rho)\star(E',\rho')$.

Calculating the product of two arrows turns out to be more complicated, and the formula is rather unpleasant, as we will now see. Given two arrows $(t,g)\colon(E_1,\rho_1)\to(E_2,\rho_2)$ and $(t',g')\colon(E'_1,\rho'_1)\to(E'_2,\rho'_2)$ in $\mathfrak{Z}(\Gamma,\mathbb{G})$, we define their product to be the arrow

$$(t\star t',gg')\colon (E_1,\rho_1)\star(E'_1,\rho'_1)\to(E_2,\rho_2)\star(E'_2,\rho'_2),$$

where $t\star t'$ is the homomorphism

$$t\star t'\colon E_1\star E'_1\to E_2\star E'_2,$$

$$(x,y)\mapsto\big(\{g',\rho_1(x)^{-1}g\}\{\rho'_1(y)g',{}^{\pi(x)}g^{-1}\}^{\rho_1(x)^{-1}g}\cdot t(x)\ ,\ t'(y)\big).$$

The formula takes the much simpler form of

$$(x,y)\mapsto\big(t(x),t'(y)\big)$$

in the case where $g=g'=1$. But in general it seems our formula can not be simplified further.

Finally, if we have two 2-arrows $(t_1,g_1)\Rightarrow(t_2,g_2)$ and $(t'_1,g'_1)\Rightarrow(t'_2,g'_2)$ given by $\mu,\mu'\in G_1$, their product is defined by $\mu^{g'_1}\mu'$.

7.5. **The case of a symmetric braiding.** In the case where the braiding on $\mathbb{G}$ is symmetric, $\mathfrak{Z}(\Gamma,\mathbb{G})$ inherits a symmetric braiding

$$b_{(E,\rho),(E',\rho')}\colon (E,\rho)\star(E',\rho')\to(E',\rho')\star(E,\rho)$$

which is defined by

$$(x,y)\mapsto\big(\iota\{\rho(x)^{-1},\rho'(y)^{-1}\}y\ ,\ x\big).$$

This braiding is symmetric in the sense that

$$b_{(E,\rho),(E',\rho')}\circ b_{(E',\rho'),(E,\rho')}=\mathrm{id}_{(E,\rho),(E',\rho')}\,.$$

Since $\mathfrak{Z}(\Gamma,\mathbb{G})$ is a group object in *2-groupoids*, there is one more piece of data that goes into the definition of a braiding on it. Given two arrows $(t,g)\colon(E_1,\rho_1)\to(E_2,\rho_2)$ and $(t',g')\colon(E'_1,\rho'_1)\to(E'_2,\rho'_2)$ in $\mathfrak{Z}(\Gamma,\mathbb{G})$, we need a 2-arrow ψ making the following diagram commute. (To make the diagram less involved, we abbreviate (E,ρ) to E.)

$$\begin{array}{ccc}
E_1\star E'_1 & \xrightarrow{b_{E_1,E'_1}} & E'_1\star E_1\\
{\scriptstyle (t,g)\star(t'.g')}\downarrow & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle (t',g')\star(t,g)}\\
E_2\star E'_2 & \xrightarrow[b_{E_2,E'_2}]{} & E'_2\star E_2
\end{array}$$

We take ψ to be $\{g,g'\}$.

As we pointed out in §7.3, the multiplication in $\mathfrak{Z}(\Gamma,\mathbb{G})$ makes it into a group object in the category of 2-groupoids and weak functors, and the corresponding 2-crossed-module is equivalent to $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$. The above discussion can be summarized by saying that, when $\mathbb{G}$ is symmetric, the 2-crossed-module $\mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$ is braided and "symmetric". (We use quotes because we are not aware of a precise definition of the notion of symmetric braided 2-crossed-module.)

The same discussion applies to the 2-group obtained by identifying 2-isomorphic arrows in $\mathfrak{Z}(\Gamma,\mathbb{G})$. Thus, the crossed-module $[d\colon C^0(\Gamma,\mathbb{G})/B^0(\Gamma,\mathbb{G})\to Z^1(\Gamma,\mathbb{G})]$ introduced in §4.1 inherits a symmetric braiding. We have already encountered this braiding in Lemma 4.4.

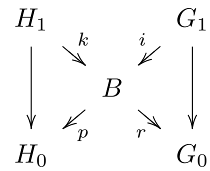
8. Functoriality of $\mathfrak{Z}(\Gamma,\mathbb{G})$

In this section the reader is assumed to have some basic familiarity with the formalism of butterflies [Noo2]. One advantage of working with butterflies, as opposed to strict morphisms of crossed-modules, is that certain calculations become completely categorical and simple. Another advantage is that questions regarding invariance under equivalence of crossed-modules get automatically taken care of.

The main result of this section can be summarized by saying that $\mathfrak{Z}(\Gamma,\mathbb{G})$ is functorial with respect to weak Γ-equivariant morphisms $B\colon\mathbb{G}\to\mathbb{H}$ of crossed-modules (read strong Γ-equivariant butterflies).

8.1. **Strong Γ-butterflies.** We begin by recalling the definition of a strong butterfly ([AlNo1], Definition 4.1.6).

Definition 8.1. A **strong butterfly** $(B, s)\colon \mathbb{H} \to \mathbb{G}$ consists of a butterfly

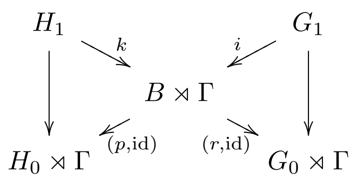


together with a set theoretic section $s\colon H_0 \to B$ for p. When $\mathbb{G}$ and $\mathbb{H}$ carry a strict Γ-action, a **Γ-butterfly** is a butterfly for which the group B is endowed with a Γ-action such that the four maps i, k, p and r are Γ-equivariant. A **strong Γ-butterfly** is a Γ-butterfly whose underlying butterfly is strong. A morphism of strong Γ-butterflies is a morphism of the underlying butterflies (§6.1) in which the homomorphism $t\colon E \to E'$ is Γ-equivariant. Finally, the definition of a 2-morphism is the one which ignores the section s and the Γ-action.

Remark 8.2. Under the correspondence between butterflies and weak morphisms, Γ-butterflies correspond to *weakly* Γ-equivariant weak morphisms.

With the composition defined as in ([Noo2], §10.1), strong Γ-butterflies form a bicategory which is biequivalent to the bicategory of Γ-butterflies (via the forgetful functor forgetting the section).

A Γ-butterfly as in Definition 8.1 gives rise to a butterfly



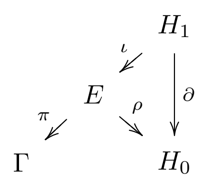
If B is strong, then this butterfly is also strong in a natural way. This construction respects composition of (strong) Γ-butterflies. More precisely, it gives rise to a trifunctor from the tricategory of Γ-crossed-modules and Γ-butterflies to the tricategory of crossed-modules and butterflies.

8.2. **Functoriality of $\mathfrak{Z}(\Gamma, \mathbb{G})$.** The 2-group $\mathfrak{Z}(\Gamma, \mathbb{G})$ is functorial in the second variable in the following sense: for a fixed Γ, $\mathfrak{Z}(\Gamma, -)$ is a trifunctor from the tricategory of Γ-crossed-modules and strong Γ-butterflies to the tricategory of 2-groupoids. We will not give a detailed proof of this statement. We will only describe the effect

$$(B, s)_*\colon \mathfrak{Z}(\Gamma, \mathbb{H}) \to \mathfrak{Z}(\Gamma, \mathbb{G})$$

of a strong Γ-butterfly (B, s) from $\mathbb{H}$ to $\mathbb{G}$. The effects of morphisms and 2-arrows of strong Γ-butterflies are easy to describe.

Let $(B, s)\colon \mathbb{H} \to \mathbb{G}$ be a strong Γ-butterfly. Let (E, ρ) be an object in $\mathfrak{Z}(\Gamma, \mathbb{H})$, as in the diagram



We define the image under (B, s) of (E, ρ) in $\mathfrak{Z}(\Gamma, \mathbb{G})$ to be the pair (F, λ) which is defined as follows. Consider the fiber product $K := E \times_{\rho, H_0, p} B$. This can be made into a group by defining the product to be

$$(x, b) \cdot (y, c) := (xy\ ,\ b \cdot {}^{\pi(x)}c).$$

There is a subgroup N of this group consisting of elements of the form $\big(\iota(\alpha), k(\alpha)\big)$, $\alpha \in G_1$. We define F to be K/N. It fits in the following diagram:

$$\begin{array}{ccccc} & & & & G_1 \\ & & & \swarrow^{(1,i)} & \\ & & F & & \downarrow{\scriptstyle \partial} \\ & \swarrow^{\pi\circ\mathrm{pr}_1} & & \searrow^{\lambda} & \\ \Gamma & & & & G_0 \end{array}$$

The crossed homomorphism $\lambda \colon F \to G_0$ is given by $(x, b) \mapsto r(b)$. It is easy to verify that (F, λ) is an object in $\mathfrak{Z}(\Gamma, \mathbb{G})$.

The effect of (B, s) on an arrow $(t, h) \colon (E, \rho) \to (E', \rho')$ is the pair $\big(u, rs(g)\big)$, where $u \colon F \to F'$ is the homomorphism induced from the map

$$E \times_{\rho', H_0, p} B \to E' \times_{\rho', H_0, p}, B$$

$$(x, b) \mapsto \big(t(x)\ ,\ s(h)^{-1} \cdot b \cdot {}^{\pi(x)}s(h)\big).$$

Finally, the effect of (B, s) on a 2-arrow $\mu \colon (t, h) \Rightarrow (t', h')$, where $\mu \in H_1$, is defined to be the unique element $\nu \in G_1$ such that $i(\nu) = s(g)^{-1} s(g\partial\mu)\kappa(\mu)^{-1}$.

Remark 8.3. The functoriality of $\mathfrak{Z}(\Gamma, \mathbb{H})$ implies immediately that for every Γ-equivariant equivalence $f \colon \mathbb{H} \to \mathbb{G}$ of crossed-modules, the induced bifunctor

$$f_* \colon \mathfrak{Z}(\Gamma, \mathbb{H}) \to \mathfrak{Z}(\Gamma, \mathbb{G})$$

is a biequivalence. Therefore, the induced morphism of crossed-modules in groupoids

$$f_* \colon \mathcal{K}^{\leq 1}(\Gamma, \mathbb{H}) \to \mathcal{K}^{\leq 1}(\Gamma, \mathbb{G})$$

is an equivalence (compare Proposition 5.6). In particular, the induced maps on cohomology H^i, $i = -1, 0, 1$, are isomorphisms.

8.3. **In the presence of a braiding.**

Definition 8.4. A Γ-butterfly

$$\begin{array}{ccccc} H_1 & & & & G_1 \\ & \searrow^{k} & & \swarrow^{i} & \\ \downarrow & & B & & \downarrow \\ & \swarrow_{p} & & \searrow_{r} & \\ H_0 & & & & G_0 \end{array}$$

is **braided** if it satisfies the identity

$$k\{p(b), p(c)\}_{\mathbb{H}} \cdot i\{r(b), r(c)\}_{\mathbb{G}} = b^{-1}c^{-1}bc$$

for every $b, c \in B$. A **strong braided Γ-butterfly** is a braided Γ-butterfly together with a set theoretic section $s \colon H_0 \to E$ for p. Morphisms and 2-morphisms of strong braided Γ-butterflies are defined to be the ones of the underlying Γ-butterflies

If $\mathbb{G}$ and $\mathbb{H}$ are endowed with a Γ-equivariant braiding and B is a braided Γ-butterfly in the sense of Definition 8.4, then the bifunctor

$$(B,s)_*\colon \mathfrak{Z}(\Gamma,\mathbb{H}) \to \mathfrak{Z}(\Gamma,\mathbb{G})$$

is monoidal. The monoidal structure on this functor is given by the natural isomorphisms

$$F_{E,E'}\colon B_*(E)\star B_*(E') \to B_*(E\star E'),$$
$$\big((x,b),(y,c)\big) \mapsto (x,y,bc).$$

Here we have abbreviated $(B,s)_*$ to B_* and (E,ρ) to E. (The proof that this map is a group homomorphism is quite nontrivial and involves some lengthy calculations.) Also, given two arrows $(t,h)\colon (E_1,\rho_1)\to(E_2,\rho_2)$ and $(t',h')\colon (E'_1,\rho'_1)\to(E'_2,\rho'_2)$ in $\mathfrak{Z}(\Gamma,\mathbb{H})$, we have the following commutative 2-cell in $\mathfrak{Z}(\Gamma,\mathbb{H})$:

$$\begin{array}{ccc}
B_*(E_1)\star B_*(E'_1) & \xrightarrow{F_{E_1,E'_1}} & B_*(E'_1\star E_1) \\
{\scriptstyle B_*(t',h')\star B_*(t,h)}\big\downarrow & \overset{\varepsilon(h,h')}{\swarrow\!\!\!\swarrow} & \big\downarrow{\scriptstyle B_*\left((t,h)\star(t',h')\right)} \\
B_*(E_2)\star B_*(E'_2) & \xrightarrow[F_{E_2,E'_2}]{} & B_*(E'_2\star E_2)
\end{array}$$

where $\varepsilon(h,h')\in G_1$ is the unique element in G_1 satisfying the identity $i\varepsilon(h,h') = s(hh')^{-1}s(h)s(h')$.

Remark 8.5. It can be shown that, for a fixed Γ, $\mathfrak{Z}(\Gamma,-)$ is a trifunctor from the tricategory of braided Γ-crossed-modules and braided strong Γ-butterflies to the tricategory of monoidal 2-groupoids. We will not prove this here.

It follows from the above discussion that if $(B,s)\colon \mathbb{H}\to\mathbb{G}$ is a braided strong Γ-butterfly, then the induced weak morphism

$$(B,s)_*\colon \mathcal{Z}(\Gamma,\mathbb{H}) \to \mathcal{Z}(\Gamma,\mathbb{G})$$

of 2-groups is braided. This implies that the induced map

$$\mathcal{K}^{\leq 1}(\Gamma,\mathbb{H}) \to \mathcal{K}^{\leq 1}(\Gamma,\mathbb{G})$$

is a morphism of 2-crossed-modules. In particular, the induced map

$$H^1(\Gamma,\mathbb{H}) \to H^1(\Gamma,\mathbb{G})$$

is a group homomorphism.

9. Everything over a Grothendieck site

The discussion of §8 is valid over any Grothendieck site, but some changes need to be made in the definition of $\mathcal{Z}(\Gamma,\mathbb{G})$. We discuss this in this section and prepare the ground to compare our definition of H^i with the standard one in terms of gerbes.

Let X be a fixed Grothendieck site. By a group we mean a sheaf of groups over X. A short exact sequence means a short exact sequence of sheaves of groups.

Let $\mathbb{G}=[G_1\to G_0]$ be a crossed-module over X, and Γ a group over X acting strictly on $\mathbb{G}$. We would like to define the analog of the 2-groupoid $\mathfrak{Z}(\Gamma,\mathbb{G})$. The definition is more or less the same as in the discrete case, with one slight change in the definition of arrows (hence, also of 2-arrows). For this reason, we will use a different notation $\mathfrak{Z}'(\Gamma, \mathbb{G})$ for it.

The crossed-module $\mathbb{G} = [\partial\colon G_1 \to G_0]$ gives rise to a quotient stack $[G_0/G_1]$ where G_1 acts on G_0 by right multiplication via ∂. That is, $[G_0/G_1]$ is the quotient stack of the transformation groupoid $[G_0 \ltimes G_1 \rightrightarrows G_0]$. The latter is a strict group object in the category of groupoids. Therefore, the quotient stack $[G_0/G_1]$ is in fact a group stack. We denote this group stack by $\mathcal{G}$.

9.1. **A provisional definition in terms of group stacks.** We include this subsection just to motivate the definition of $\mathfrak{Z}'(\Gamma, \mathbb{G})$ that will be given in §9.2. Using the idea discussed in Remark 7.1, we introduce a closely related (and naturally biequivalent) 2-groupoid which is defined in terms of group stacks. This 2-groupoid, though conceptually much simpler, is not very explicit. In §9.2 we use the results of [AlNo1] to translate this definition to the language of crossed-modules.

Let $\mathcal{G}$ be a group stack (for example, the quotient stack of $\mathbb{G}$) with an action of a group Γ. The associated 2-groupoid is defined as follows.

An object in this 2-groupoid is a (weak) morphism of group stacks $r\colon \Gamma \to \mathcal{G} \rtimes \Gamma$ such that $\mathrm{pr}_2 \circ r = \mathrm{id}_\Gamma$.

A morphism $(t, g)\colon r \to r'$ in this 2-groupoid consists of a global section g of $\mathcal{G}$ and a monoidal transformation $t\colon r \to gr'g^{-1}$, where $gr'g^{-1}$ is the morphism r' composed with the conjugation by g automorphism of $\mathcal{G}$. The composition of two morphisms (t, g) and (t', g') is defined to be $\big(t(gt'g^{-1}), gg'\big)$.

A 2-arrow $\mu\colon (t, g) \Rightarrow (t', g')$ is a transformation $\mu\colon g \to g'$ which intertwines t and t'.

9.2. **The 2-groupoid $\mathfrak{Z}'(\Gamma, \mathbb{G})$.** Thanks to the equivalence of butterflies and weak morphisms of group stacks [AlNo1], we can translate the definition given in §9.1 and find a more convenient definition of $\mathfrak{Z}'(\Gamma, \mathbb{G})$ along the lines of §7.

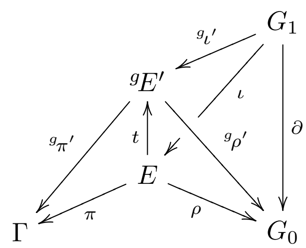
Some notation. We use the notation $X \overset{G}{\times} Y$ for the contracted product of two sets X and Y with an action of a group G. Breen (and also [AlNo1]) uses the notation $X \overset{G}{\wedge} Y$. If X and Y are over a third set Z and the G-actions are fiberwise, we denote by $X \underset{Z}{\overset{G}{\times}} Y$ the subset in $X \overset{G}{\times} Y$ consisting of those pairs (x, y) such that x and y map to the same element in Z.

Objects of $\mathfrak{Z}'(\Gamma, \mathbb{G})$. The objects of $\mathfrak{Z}'(\Gamma, \mathbb{G})$ turn out to be exactly the same as before. Namely, they are diagrams

$$\begin{array}{ccccc} & & & & G_1 \\ & & & \overset{\iota}{\swarrow} & \\ & & E & & \Big\downarrow \partial \\ & \overset{\pi}{\swarrow} & & \overset{\rho}{\searrow} & \\ \Gamma & & & & G_0 \end{array}$$

of sheaves of groups over X such that the diagonal sequence is short exact and ρ is a crossed-homomorphism intertwining the conjugation action of E on G_1 with the crossed-module action of G_0 on G_1 (see §7.1).

Arrows of $\mathfrak{Z}'(\Gamma, \mathbb{G})$. An arrow in $\mathfrak{Z}'(\Gamma, \mathbb{G})$ from (E, ρ) to (E', ρ') is a pair (t, g) where g and t are as follows. The g here is a pair (P, φ), where P is a right G_1-torsor on X and $\varphi\colon P \to G_0$ is a G_1-equivariant morphism of sheaves. Here G_1 acts on G_0 by right multiplication via ∂. The t is an isomorphism $E \to {}^{g}E'$ of sheaves of groups making the following diagram commute



Here, ${}^{g}E' := P \overset{G_1}{\times} E'$ is the contracted product of P and E', where G_1 acts on E' by right conjugation. The map ${}^{g}\pi'\colon P \overset{G_1}{\times} E' \to \Gamma$ is $\pi' \circ \mathrm{pr}_2$. The map ${}^{g}\rho'$ is defined by

$$
\begin{aligned}
{}^{g}\rho'\colon P \overset{G_1}{\times} E' &\to G_0 \\
(u, x) &\mapsto \varphi(u) \cdot \rho'(x) \cdot {}^{\pi'(x)}\varphi(u)^{-1},
\end{aligned}
$$

and ${}^{g}\iota'$ is defined by

$$
\begin{aligned}
{}^{g}\iota'\colon G_1 &\to P \overset{G_1}{\times} E' \\
\alpha &\mapsto \big(u, \iota'(\alpha^{\varphi(u)})\big),
\end{aligned}
$$

where $u \in P$ is randomly chosen; it is easy to see that the pair $\big(u, \iota'(\alpha^{\varphi(u)})\big)$, viewed as an element in $P \overset{G_1}{\times} E'$, is independent of u.

Remark 9.1. The object $({}^{g}E, {}^{g}\rho)$ should be regarded as the left conjugate of (E, ρ) under the action of g. Note that, by definition of the quotient stack, $g = (P, \varphi)$ is a global section of $\mathcal{G} = [G_0/G_1]$.

The composition of two arrows $(t, g)\colon (E, \rho) \to (E', \rho')$ and $(t', g')\colon (E', \rho') \to (E'', \rho'')$ is defined to be (t'', g''), where g'' and t'' are defined as follows. First we define g''. Let $g = (P, \varphi)$ and $g' = (P', \varphi')$. Make P' into a left G_1-torsor (indeed, a bitorsor) by setting

$$
{}^{\alpha}u := u^{\alpha^{\varphi(u)}}, \ \alpha \in G_1, u \in P'.
$$

Form the contracted product $P \overset{G_1}{\times} P'$, where now P' is viewed as a left G_1-torsor. It inherits a right G_1-torsor structure from P'. Define $\varphi \overset{G_1}{\times} \varphi'$ by the rule

$$
\varphi \overset{G_1}{\times} \varphi'\colon P \overset{G_1}{\times} P' \to G_0, \quad (u, v) \mapsto \varphi(u)\varphi'(v).
$$

We define g'' to be the pair $(P \overset{G_1}{\times} P', \varphi \overset{G_1}{\times} \varphi')$. (Remark that if we view g and g' as global sections of the group stack $\mathcal{G} = [G_0/G_1]$, then g'' corresponds to the product gg').

The homomorphism t'' is defined to be

$$
t''\colon E \to P \overset{G_1}{\times} P' \overset{G_1}{\times} E''
$$

$$
t'' := (P \overset{G_1}{\times} t') \circ t.
$$

2-arrows of $\mathfrak{Z}'(\Gamma,\mathbb{G})$. A 2-arrow $(t,g)\Rightarrow(t',g')$ in $\mathfrak{Z}'(\Gamma,\mathbb{G})$ is an isomorphism $\mu\colon g\to g'$ such that the diagram

$$\begin{array}{ccc} {}^{g}E & \xrightarrow{\mu \overset{G_1}{\times} E} & {}^{g'}E \\ & {\scriptstyle t}\searrow \qquad \swarrow{\scriptstyle t'} & \\ & E & \end{array}$$

commutes. Here, by an isomorphism $\mu\colon g\to g'$ we mean an isomorphism $P_g\to P_{g'}$ of G_1-torsors, which we denote again by μ, making the diagram

$$\begin{array}{ccc} P_g & \xrightarrow{\mu} & P_{g'} \\ & {\scriptstyle \varphi}\searrow \qquad \swarrow{\scriptstyle \varphi'} & \\ & G_0 & \end{array}$$

commute.

Remark 9.2. In contrast with $\mathfrak{Z}(\Gamma,\mathbb{G})$ which is a 2-groupoid, $\mathfrak{Z}'(\Gamma,\mathbb{G})$ is a bigroupoid.

9.3. **Functoriality of $\mathfrak{Z}'(\Gamma,\mathbb{G})$.** The bigroupoid $\mathfrak{Z}'(\Gamma,\mathbb{G})$ is in some sense more natural than $\mathfrak{Z}(\Gamma,\mathbb{G})$, because it is actually functorial with respect to Γ-butterflies. That is, we do not need strong butterflies (Definition 8.1) in order to define pushforwards.

Let $B\colon\mathbb{H}\to\mathbb{G}$ be a Γ-butterfly

$$\begin{array}{ccccc} H_1 & & & & G_1\\ \Big\downarrow & \overset{k}{\searrow} & & \overset{i}{\swarrow} & \Big\downarrow \\ & & B & & \\ \Big\downarrow & \underset{p}{\swarrow} & & \underset{r}{\searrow} & \Big\downarrow\\ H_0 &&&& G_0\end{array}$$

The bifunctor $B_*\colon\mathfrak{Z}'(\Gamma,\mathbb{H})\to\mathfrak{Z}'(\Gamma,\mathbb{G})$ is defined as follows.

Effect of B_* on objects. Let (E,ρ) be an object in $\mathfrak{Z}'(\Gamma,\mathbb{H})$. The effect of B_* is given by

$$\begin{array}{ccccc} & & H_1 & & \\ & \overset{\iota}{\swarrow} & \Big\downarrow & & \\ & E & \Big\downarrow{\scriptstyle \partial} & & \\ \overset{\pi}{\swarrow} & & \overset{\rho}{\searrow}\Big\downarrow & & \\ \Gamma & & H_0 & & \end{array} \quad \overset{B_*}{\longmapsto} \quad \begin{array}{ccccc} & & G_1 & & \\ & \swarrow & \Big\downarrow & & \\ & F & \Big\downarrow{\scriptstyle \partial} & & \\ \swarrow & & \overset{\lambda}{\searrow}\Big\downarrow & & \\ \Gamma & & G_0 & & \end{array}$$

where $B_*(E,\rho)$ is defined exactly as in §8.2. Namely, it is equal to (F,λ) with

$$F:=E\overset{H_1}{\underset{H_0}{\times}}B \quad\text{and}\quad \lambda\colon F\to G_0,\ (x,b)\mapsto r(b).$$

Here, G_1 acts on each component by right multiplication.

Effect of B_* on arrows. Let $(t,h)\colon(E,\rho)\to(E',\rho')$ be an arrow in $\mathfrak{Z}'(\Gamma,\mathbb{H})$. Here, h is equal to (P,φ), where P is an H_1-torsor and $\varphi\colon P\to H_0$ is an H_1-equivariant map, and $t\colon E \to P \overset{H_1}{\times} E'$ is a homomorphism. We define $B_*(t,h)$ to be (s,g), where g and s are defined as follows.

Consider $Q := P \overset{H_1}{\underset{H_0}{\times}} B$, where H_1 acts on B by right multiplication via k. Since the images of k and i in B commute, the right multiplication action of G_1 on B via i gives rise to a right action of G_1 on Q. It is easy that this makes Q into a right G_1-torsor. We have a G_1-equivariant map

$$\chi\colon Q = P \overset{H_1}{\underset{H_0}{\times}} B \to G_0$$
$$(x,b) \mapsto r(b).$$

We define g to be (Q,χ). The homomorphism $s\colon F \to {}^gF'$ is defined to be the composition

$$F = E \overset{H_1}{\underset{H_0}{\times}} B \xrightarrow{t \overset{H_1}{\underset{H_0}{\times}} B} (P \overset{H_1}{\times} E') \overset{H_1}{\underset{H_0}{\times}} B \xrightarrow{\eta^{-1}} (P \overset{H_1}{\underset{H_0}{\times}} B) \overset{G_1}{\times} (E' \overset{H_1}{\underset{H_0}{\times}} B) = Q \overset{G_1}{\times} F' = {}^gF'$$

For the convenience of the reader, let us clarify all the actions appearing in the above expression, as well as define the isomorphism η.

In $(P \overset{H_1}{\times} E') \overset{H_1}{\underset{H_0}{\times}} B$, the action of the first H_1 on E' is by right conjugation, and the action of the second H_1 on B is by right multiplication. The action of the second H_1 on $P \overset{H_1}{\times} E'$ is by right multiplication via ${}^g\iota'$. That is, (u,x) acted on by $\alpha \in H_1$ is equal to $(u, x\alpha^{\varphi(u)})$.

In $(P \overset{H_1}{\underset{H_0}{\times}} B) \overset{G_1}{\times} (E' \overset{H_1}{\underset{H_0}{\times}} B)$ all actions are by right multiplication, except for the action of G_1 on the last B component which is by right conjugation.

Finally, the isomorphism η is defined by

$$\eta\colon (P \overset{H_1}{\underset{H_0}{\times}} B) \overset{G_1}{\times} (E' \overset{H_1}{\underset{H_0}{\times}} B) \;\to\; (P \overset{H_1}{\times} E') \overset{H_1}{\underset{H_0}{\times}} B$$
$$(u,b,y,c) \;\mapsto\; (u,y,bcb^{-1}).$$

We leave it to the reader to verify that this is indeed an isomorphism of groups.

Effect of B_* on 2-arrows. This is defined in the obvious way.

9.4. **Comparing $\mathfrak{Z}'(\Gamma,\mathbb{G})$ and $\mathfrak{Z}(\Gamma,\mathbb{G})$.** Instead of defining $\mathfrak{Z}'(\Gamma,\mathbb{G})$ as in §9.2, we could have imitated the definition of $\mathfrak{Z}(\Gamma,\mathbb{G})$ given in §7.1. We argue that this would not have been the correct definition. Let us analyze what goes wrong with this naive definition. There is a natural bifunctor

$$\Psi\colon \mathfrak{Z}(\Gamma,\mathbb{G}) \to \mathfrak{Z}'(\Gamma,\mathbb{G})$$

which is the identity on objects and is fully faithful on hom groupoids. This functor, however, misses many arrows in $\mathfrak{Z}'(\Gamma,\mathbb{G})$. This is essentially because not every global section of the quotient stack $[G_0/G_1]$ lifts to a global section of G_0. Let us spell this out in more detail.

The functor Ψ sends an arrow (t,g) in $\mathfrak{Z}(\Gamma,\mathbb{G})$, where $g \in G_0$ and $t\colon E \to E'$ is a group homomorphism (with certain properties), to a pair $(\hat{g},\hat{t})$ in which $\hat{g}$ is the pair (G_1,φ) with G_1 the trivial G_1-torsor and $\varphi\colon G_1 \to G_0$ given by $\alpha \mapsto g\partial(\alpha)$. It follows that if an arrow in $\mathfrak{Z}'(\Gamma,\mathbb{G})$ is in the image of Ψ, or is 2-isomorphic to such an arrow, then its corresponding G_1-torsor P is trivial. The converse is also easily seen to be true.

Proposition 9.3. *There is a natural bifunctor*

$$\Psi\colon \mathfrak{Z}(\Gamma, \mathbb{G}) \to \mathfrak{Z}'(\Gamma, \mathbb{G})$$

which is the identity on objects and is fully faithful on hom groupoids. If $H^1(X, G_1)$ is trivial, then Ψ is a biequivalence. In particular, in the case where everything is discrete (i.e., X is a point), Ψ is a biequivalence.

To end this subsection, let us also recall two other differences between $\mathfrak{Z}(\Gamma, \mathbb{G})$ and $\mathfrak{Z}'(\Gamma, \mathbb{G})$. The former is a 2-groupoid and it is functorial only with respect to *strong* Γ-butterflies. The latter is a bigroupoid and is functorial with respect to all Γ-butterflies.

9.5. **Continuous, differentiable, algebraic, etc., settings.** The cocycle approach to cohomology discussed in §3-5 has the disadvantage that it is only appropriate in the discrete setting. For instance, in the case where Γ and $\mathbb{G}$ are Lie, both the differentiable and discrete cocycles give the wrong cohomologies in general.

The butterfly approach, however, always gives the correct answer. Let us elaborate this a little bit. For example, suppose that M is a manifold, Γ is a Lie group bundle over M, and $\mathbb{G} = [G_1 \to G_0]$ a bundle of Lie crossed-modules. In this case, an element in $H^1(\Gamma, \mathbb{G})$ is a diagram

$$\begin{array}{ccccc} & & & & G_1 \\ & & & \overset{\iota}{\swarrow} & \big\downarrow \\ & & E & & \\ & \overset{\pi}{\swarrow} & & \overset{\rho}{\searrow} & \big\downarrow \\ \Gamma & & & & G_0 \end{array}$$

as in §9.2 in which E is a Lie group bundle over M, the diagonal sequence is short exact in the category of Lie group bundles, and the map ρ is differentiable. Two such diagrams (E, ρ) and (E', ρ') give rise to the same cohomology class in $H^1(\Gamma, \mathbb{G})$ if and only if there exists a principal G_1-bundle P over M, a G_1-equivariant differentiable map of bundles $\varphi\colon E \to G_0$, and an isomorphism of Lie group bundles

$$f\colon P \overset{G_1}{\times} E' \to E$$

such that:

- for every $u \in P$ and $y \in E'$, $\rho f(u, y) \cdot {}^{\pi'(y)}\varphi(u) = \varphi(u) \cdot \rho'(y)$,
- for every $u \in P$ and $y \in E'$, $\pi f(u, y) = \pi'(y)$,
- for every $u \in P$ and $\alpha \in G_1$, $f(u, \iota'(\alpha^{\varphi(u)})) = \iota(\alpha)$.

Notice, in particular, that in the case where M is a point, the Lie group E and the extension

$$1 \to G_1 \to E \to \Gamma \to 1$$

are uniquely determined (up to isomorphism) by the the given element in $H^1(\Gamma, \mathbb{G})$ and can be thought as invariants of the given cohomology class.

The same discussion is valid in the algebraic setting (where G is a group scheme, or an algebraic group, and $\mathbb{G}$ is a crossed-module in group schemes, or algebraic groups), or in the topological setting, etc.

10. H^i AND GERBES

In this section, we give an interpretation of the 2-groupoid $\mathfrak{Z}'(\Gamma, \mathbb{H})$ in terms of gerbes over the classifying stack $B\Gamma$, and clarify the relation between our definition of H^i and the standard one in terms of gerbes.

Our set up is as follows. We fix a Grothendieck site X. When working over the site X, by a group we mean a sheaf of groups on X, and by a crossed-module we mean a crossed-module in sheaves of groups.

Given a sheaf of groups Γ over X, we denote the classifying stack of Γ by $B\Gamma := [\Gamma\backslash X]$. We sometimes use the same notation for the Grothendieck site $(X \downarrow B\Gamma)$ of objects in X over $B\Gamma$.

Recall that to a crossed-module $\mathbb{G} = [\partial\colon G_1 \to G_0]$ over X we can associate a group stack $\mathcal{G}$ which is, by definition, the quotient stack of the transformation groupoid $[G_0 \ltimes G_1 \rightrightarrows G_0]$. Note that the latter is a strict group object in the category of groupoids.

Our notational convention is that whenever we use the notation $[G_0/G_1]$, we simply mean the quotient stack without the group structure. When we want to take into account the group structure, we use $\mathcal{G}$. For example, we will be considering $[G_0/G_1]$ as a trivial right $\mathcal{G}$-torsor.

10.1. **Cohomology via gerbes.** It is well-known that, for every group stack $\mathcal{G}$ over a Grothendieck site X, $H^i(X, \mathcal{G})$, $i = -1, 0, 1$ are defined as follows:

- $H^{-1}(X, \mathcal{G})$ is the group of self-equivalences of the identity section of $\mathcal{G}$; this is an abelian group.
- $H^0(X, \mathcal{G})$ is the group of global sections of $\mathcal{G}$ modulo transformation; this is a group, not necessarily abelian.
- $H^1(X, \mathcal{G})$ is the set of isomorphism classes of (right) $\mathcal{G}$-torsors over X; this is a pointed set.

A Γ-crossed-module $\mathbb{G}$ gives rise to a crossed-module $\mathbb{G}_\Gamma$, and the corresponding group stack $\mathcal{G}_\Gamma$, on the classifying stack $B\Gamma$. In the case where X is a point, it is straightforward (but rather tedious) to see that we have natural isomorphisms

$$H^i(\Gamma, \mathbb{G}) \cong H^i(B\Gamma, \mathcal{G}_\Gamma), \quad i = -1, 0, 1.$$

In fact, our definitions of $H^i(\Gamma, \mathbb{G})$ given in §3 and §4 were obtained by translating the definition of $H^i(B\Gamma, \mathcal{G}_\Gamma)$ to the cocycle language. (The idea is to write down the descent data for a $\mathcal{G}_\Gamma$-torsor on $B\Gamma$ and see that we obtain the cocycles of §3 and §4.)

The right $\mathcal{G}_\Gamma$-torsors over $B\Gamma$ form a strict 2-groupoid. The morphisms of this groupoid are morphisms of $\mathcal{G}_\Gamma$-torsors, and the 2-arrows of it are transformations. Let $\mathfrak{Z}(\Gamma, \mathcal{G})$ be the full sub 2-groupoid of this 2-groupoid consisting of those $\mathcal{G}_\Gamma$-torsors which become isomorphic to the trivial $\mathcal{G}$-torsor when pulled back to X via the quotient map $X \to B\Gamma$. (We do not fix the trivialization.)

Proposition 10.1. *Let $\mathfrak{Z}(\Gamma, \mathbb{G})$ be as above and $\mathfrak{Z}'(\Gamma, \mathbb{G})$ as in §9.2. Then, there is a biequivalence*

$$\Upsilon\colon \mathfrak{Z}'(\Gamma, \mathbb{G}) \to \mathfrak{Z}(\Gamma, \mathcal{G})$$

which is natural up to higher coherences.

Proof. We give an outline of the construction of this biequivalence.

Effect of Υ *on objects.* Let (E,ρ) be an object in $\mathfrak{Z}'(\Gamma,\mathbb{G})$, as in the diagram

$$\begin{array}{ccccc} & & & & G_1 \\ & & & \swarrow^{\iota} & \\ & & E & & \Big\downarrow \partial \\ & \swarrow^{\pi} & & \searrow^{\rho} & \\ \Gamma & & & & G_0 \end{array}$$

To this we want to associate a right $\mathcal{G}_\Gamma$-torsor over $B\Gamma$. Think of $[\iota\colon G_1 \to E]$ as a crossed-module (via the conjugation action of E on G_1), and let $\Gamma' := [E/G_1]$ be the corresponding group stack. That is, the underlying stack of Γ' is the quotient stack of the groupoid $[E \ltimes G_1 \rightrightarrows E]$. As we pointed out at the beginning of this section, the latter is a strict group object in the category of groupoids. There is a natural equivalence of group stacks $\phi\colon \tilde{\Gamma} \to \Gamma$ induced by π.

The map ϕ provides us a left action of $\tilde{\Gamma}$ on the group stack $\mathcal{G}$ via that of Γ. We will show that there is also a natural action of $\tilde{\Gamma}$ on the stack $[G_0/G_1]$ which makes the right $\mathcal{G}$-torsor structure of $[G_0/G_1]$ $\tilde{\Gamma}$-equivariant. After *choosing* an inverse for ϕ, this gives rise to an action of Γ on the trivial $\mathcal{G}$-torsor $[G_0/G_1]$. Passing to Γ-quotients, we obtain a $\mathcal{G}_\Gamma$-torsor $\mathcal{P}$ on the classifying stack $B\Gamma$.

Let us now spell out the action of $\tilde{\Gamma}$ on the stack $[G_0/G_1]$. We do this on the groupoid level. That is, we give a left action of $[E \ltimes G_1 \rightrightarrows E]$, viewed as a group object in groupoids, on the groupoid $[G_0 \ltimes G_1 \rightrightarrows G_0]$. To do so, we give an automorphism F_x from $[E \ltimes G_1 \rightrightarrows E]$ to itself for every $x \in E$. Also, for every arrow (x,β) between the objects x and $y = x\iota(\beta)$ in $[E \ltimes G_1 \rightrightarrows E]$, we give a transformation $T_{(x,\beta)}\colon F_x \Rightarrow F_y$.

The effect of the automorphism F_x on an object $g \in G_0$ of the groupoid $[G_0 \ltimes G_1 \rightrightarrows G_0]$ is given by

$$g \;\mapsto\; \rho(x)\cdot {}^{\pi(x)}g.$$

Its effect on an arrow $(g,\alpha) \in G_0 \ltimes G_1$ is given by

$$(g,\alpha) \;\mapsto\; \big(\rho(x)\cdot {}^{\pi(x)}g \;,\; {}^{\pi(x)}\alpha\big).$$

The transformation $T_{(x,\beta)}\colon F_x \Rightarrow F_y$ is defined by

$$T_{(x,\beta)}(g) := \big(\rho(x)\cdot {}^{\pi(x)}g \;,\; \beta^{({}^{\pi(x)}g)}\big).$$

Here, $g \in G_0$ is viewed as on object and $\big(\rho(x)\cdot {}^{\pi(x)}g \;,\; \beta^{({}^{\pi(x)}g)}\big) \in G_0 \ltimes G_1$ as an arrow in the groupoid $[G_0 \ltimes G_1 \rightrightarrows G_0]$.

Effect of Υ *on arrows and 2-arrows.* Let $(t,g)\colon (E,\rho) \to (E',\rho')$ be an arrow in $\mathfrak{Z}'(\Gamma,\mathbb{G})$. Let $\bar{g}$ be the global section of $[G_0/G_1]$ over X corresponding to $g = (P,\varphi)$. It can be checked that left multiplication on $[G_0/G_1]$ by $\bar{g}$ is Γ-equivariant and it respects the right $\mathcal{G}$-torsor structures. (This is perhaps easiest to see by trivializing the G_1-torsor P over some open cover of X and then showing that the Γ-equivariance data are compatible along the intersections of the open sets.) After passing to the Γ-quotients, we obtain an equivalence of $\mathcal{G}_\Gamma$-torsors $\mathcal{P} \to \mathcal{P}'$ over $B\Gamma$. This defines the effect of Υ on morphisms.

The definition of the effect of Υ on 2-arrows is straightforward.

The inverse of Υ. Denote $B\Gamma$ by $\mathcal{Y}$ for simplicity. Let $\mathcal{P}$ be a $\mathcal{G}_\Gamma$-torsor over $\mathcal{Y}$ which becomes trivial after pulling back along the quotient map $q\colon X \to \mathcal{Y}$. *Choose* a trivialization, that is, a map $f\colon X \to \mathcal{P}$ relative to $\mathcal{Y}$. Let $\delta\colon \mathcal{P}\times_{\mathcal{Y}}\mathcal{P} \to \mathcal{G}_\Gamma$ be "the" difference map. That is, δ is a morphism such that

$$\mathcal{P}\times_{\mathcal{Y}}\mathcal{P} \stackrel{(\mathrm{pr}_1,\delta)}{\longrightarrow} \mathcal{P}\times_{\mathcal{Y}}\mathcal{G}_\Gamma$$

becomes an inverse to

$$\mathcal{P}\times_{\mathcal{Y}}\mathcal{G}_\Gamma \stackrel{(\mathrm{pr}_1,\mu)}{\longrightarrow} \mathcal{P}\times_{\mathcal{Y}}\mathcal{P},$$

where μ stands for the action. (Note that we have to make a *choice* of the inverse, and δ depend on this choice.)

Consider the morphism

$$X\times_{\mathcal{Y}} X \xrightarrow{f\times_{\mathcal{Y}} f} \mathcal{P}\times_{\mathcal{Y}}\mathcal{P} \xrightarrow{\ \delta\ } \mathcal{G}_\Gamma\ .$$

Observe that this morphism is over $\mathcal{Y}$ and that $X\times_{\mathcal{Y}} X$ is naturally equivalent to Γ. Thus, we obtain a morphism $\Gamma \to \mathcal{G}_\Gamma$ fitting in a 2-cartesian diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathcal{G}_\Gamma \\ \downarrow & & \downarrow \\ X & \xrightarrow[q]{} & \mathcal{Y} \end{array}$$

Since the pullback of $\mathcal{G}_\Gamma$ along q is naturally equivalent to $\mathcal{G}$, we obtain a morphism $\rho\colon \Gamma \to \mathcal{G}$. This map can be checked to be a crossed-homomorphism.

Now, we follow the argument of ([AlNo1], §4.2.4) and set

$$E := \Gamma \times_{\bar{\rho},\mathcal{G},q} G_0,$$

where $q\colon G_0 \to \mathcal{G}$ is the quotient map. By ([AlNo1], §4.2.5), E fits in a diagram

$$\begin{array}{ccccc} & & & & G_1 \\ & & & \iota\swarrow & \downarrow \partial \\ & & E & & \\ & \mathrm{pr}_1\swarrow & & \searrow \mathrm{pr}_2 & \\ \Gamma & & & & G_0 \end{array}$$

with the desired properties. This defines the effect of Υ^{-1} on objects.

From the above construction it is clear how to define the effect of Υ^{-1} on arrows and 2-arrows. □

10.2. **In the presence of a braiding.** In this subsection, we show that if $\mathbb{G}$ is endowed with a Γ-equivariant braiding, then there is a product on $\mathfrak{Z}(\Gamma,\mathcal{G})$ which makes it into a (weak) group object in the category of 2-groupoids. Our construction was conceived in a discussion with E. Aldrovandi and relies on the tools developed in ([AlNo1], §7), to which we refer the reader for more details.

Suppose that $\mathbb{G}$ is equipped with a Γ-equivariant braiding. In this case, the butterfly of Example 6.2 becomes Γ-equivariant. Therefore, we have a butterfly $\mathbb{G}_\Gamma\times\mathbb{G}_\Gamma \to \mathbb{G}_\Gamma$ over $B\Gamma$. This in turn gives rise to a morphism $m\colon \mathcal{G}_\Gamma\times\mathcal{G}_\Gamma \to \mathcal{G}_\Gamma$ of group stacks over $B\Gamma$. It follows that with this multiplication $\mathcal{G}_\Gamma$ is a group object in the category of group stacks over $B\Gamma$.

We can use the morphism m to define a multiplication on $\mathfrak{Z}(\Gamma,\mathcal{G})$ as follows. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be $\mathcal{G}_\Gamma$-torsors over $B\Gamma$. Then, $\mathcal{P}_1\times\mathcal{P}_2$ is a $\mathcal{G}_\Gamma\times\mathcal{G}_\Gamma$-torsor. The 'extension of structure group' functor for the map $m\colon \mathcal{G}_\Gamma\times\mathcal{G}_\Gamma \to \mathcal{G}_\Gamma$ applied to the

$\mathcal{G}_\Gamma \times \mathcal{G}_\Gamma$-torsor $\mathcal{P}_1 \times \mathcal{P}_2$ gives a $\mathcal{G}_\Gamma$-torsors $\mathcal{P}_1 \cdot \mathcal{P}_2$. This is the desired product of $\mathcal{P}_1$ and $\mathcal{P}_2$. More precisely,

$$\mathcal{P}_1 \cdot \mathcal{P}_2 := (\mathcal{P}_1 \times \mathcal{P}_2) \overset{\mathcal{G}_\Gamma \times \mathcal{G}_\Gamma}{\times} \mathcal{G}_\Gamma,$$

where the $\mathcal{G}_\Gamma$ on the right is made into a left $\mathcal{G}_\Gamma \times \mathcal{G}_\Gamma$-torsor via m.

The same construction can be used to define the product of morphisms and 2-arrows of $\mathfrak{Z}(\Gamma, \mathcal{G})$.

In the case where the braiding on $\mathbb{G}$ is symmetric, $\mathbb{G}_\Gamma$ becomes a symmetric braided crossed-module over $B\Gamma$ and the multiplication $m\colon \mathcal{G}_\Gamma \times \mathcal{G}_\Gamma \to \mathcal{G}_\Gamma$ becomes braided ([AlNo1], §7.2). That is, m and $m \circ \tau$, where $\tau\colon \mathcal{G}_\Gamma \times \mathcal{G}_\Gamma \to \mathcal{G}_\Gamma \times \mathcal{G}_\Gamma$ is the switch map, become isomorphic via a natural isomorphism satisfying the well-known coherence relations. This implies that the product on $\mathfrak{Z}(\Gamma, \mathcal{G})$ is braided. This braiding is compatible with the braiding of $\mathfrak{Z}(\Gamma, \mathbb{G})$ under the equivalence of Proposition 10.1.

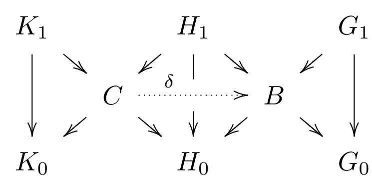
11. Cohomology long exact sequence

In this section, we show that to any short exact sequence of Γ-crossed-modules and Γ-butterflies one can associate a long exact sequence in cohomology (Proposition 11.3).

11.1. **Short exact sequences of butterflies.** Let $\mathbb{K} \xrightarrow{C} \mathbb{H}$ and $\mathbb{H} \xrightarrow{B} \mathbb{G}$ be butterflies. We say that

$$1 \to \mathbb{K} \xrightarrow{C} \mathbb{H} \xrightarrow{B} \mathbb{G} \to 1$$

is **short exact** if in the diagram



we can find an arrow $\delta\colon C \to B$ such that the diagram is commutative and the sequence

$$1 \to K_1 \longrightarrow C \xrightarrow{\delta} B \longrightarrow G_0 \to 1$$

is exact. (Note that δ is not necessarily unique.)

Example 11.1. Assume C and B are strict butterflies, that is, they come from strict morphisms $(c_1, c_0)\colon [K_1 \to K_0] \to [H_1 \to H_0]$ and $(b_1, b_0)\colon [H_1 \to H_0] \to [G_1 \to G_0]$ of crossed-modules ([Noo2], §9.5). Then, the sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_1 & \xrightarrow{c_1} & H_1 & \xrightarrow{b_1} & G_1 & \longrightarrow & 1 \\
\downarrow & & \downarrow{\scriptstyle \partial_{\mathbb{K}}} & & \downarrow{\scriptstyle \partial_{\mathbb{H}}} & & \downarrow{\scriptstyle \partial_{\mathbb{G}}} & & \downarrow \\
1 & \longrightarrow & K_0 & \xrightarrow[c_0]{} & H_0 & \xrightarrow[b_0]{} & G_0 & \longrightarrow & 1
\end{array}$$

is exact if and only if there exists a map $\psi\colon K_0 \to G_1$ such that:

- for every $k, k' \in K_0$, $\psi(kk') = \psi(k)^{b_0 c_0(k')} \psi(k')$,
- the images of $\partial_{\mathbb{G}}$ and b_0 generate G_0,

- the intersection of the kernels of $\partial_{\mathbb{K}}$ and c_1 is trivial,
- for every $k \in K_0$, $b_0c_0(k) \cdot \partial\psi(k) = 1$,
- for every $\gamma \in K_1$, $b_1c_1(\gamma) \cdot \psi(\partial\gamma) = 1$,
- if $k \in K_0$ and $\beta \in H_1$ are such that $c_0(k)\partial\beta = 1$ and $\psi(k) = b_1(\beta)$, then there exists $\gamma \in K_1$ such that $k = \partial\gamma$ and $\beta = c_1(\gamma)^{-1}$,
- if $h \in H_0$ and $\alpha \in G_1$ are such that $b_0(h)\partial\alpha = 1$, then there exist $k \in K_0$ and $\beta \in H_1$ such that $h = c_0(k)\partial\beta$ and $\alpha = b_1(\beta)^{-1}\psi(k)$.

Observe that the above list of conditions is equivalent to the sequence

$$1 \to K_1 \to K_0 \ltimes H_1 \to H_0 \ltimes G_1 \to G_0 \to 1$$

being exact. The maps in this sequence are as follows:

$$K_1 \to K_0 \ltimes H_1,\ \gamma \mapsto (\partial\gamma\,,\, c_1(\gamma^{-1}));$$
$$K_0 \ltimes H_1 \to H_0 \ltimes G_1,\quad (k,\beta) \mapsto \big(c_0(k)\partial\beta\,,\, b_1(\beta)^{-1}\psi(k)\big);$$
$$H_0 \ltimes G_1 \to G_0,\ (h,\alpha) \mapsto b_0(h)\partial(\alpha).$$

The proof of the following proposition will appear in [Noo3].

Proposition 11.2. *A sequence of crossed-modules and butterflies*

$$1 \to \mathbb{K} \xrightarrow{C} \mathbb{H} \xrightarrow{B} \mathbb{G} \to 1$$

is exact if and only if the induced sequence

$$1 \to \mathcal{K} \xrightarrow{C} \mathcal{H} \xrightarrow{B} \mathcal{G} \to 1$$

of group stacks is exact in the sense of ([AlNo1], §*6.2).*

11.2. **Cohomology long exact sequence.** By applying ([AlNo1], Proposition 6.4.1) to the site $B\Gamma$ and making use of Proposition 10.1, we immediately obtain the following (also see [CeFe], Theorem 31).

Proposition 11.3. *Let*

$$1 \to \mathbb{K} \xrightarrow{C} \mathbb{H} \xrightarrow{B} \mathbb{G} \to 1$$

be a short exact sequence of Γ*-crossed-modules and* Γ*-butterflies. Then, we have a long exact cohomology sequence*

$$1 \to H^{-1}(\Gamma,\mathbb{K}) \to H^{-1}(\Gamma,\mathbb{H}) \to H^{-1}(\Gamma,\mathbb{G}) \to H^0(\Gamma,\mathbb{K}) \to$$
$$H^0(\Gamma,\mathbb{H}) \to H^0(\Gamma,\mathbb{G}) \to H^1(\Gamma,\mathbb{K}) \to H^1(\Gamma,\mathbb{H}) \to H^1(\Gamma,\mathbb{G}).$$

(Remark that the connecting homomorphisms in this long exact sequence depend on the choice of the homomorphism δ *appearing in the definition of a short exact sequence.)*

The above proposition can be strengthened as follows (see [CeFe], Proposition 30).

Proposition 11.4. *Let*

$$1 \to \mathbb{K} \xrightarrow{C} \mathbb{H} \xrightarrow{B} \mathbb{G} \to 1$$

be a short exact sequence of Γ*-crossed-modules and* Γ*-butterflies. Then, the sequence*

$$\mathfrak{Z}'(\Gamma,\mathbb{K}) \xrightarrow{C_*} \mathfrak{Z}'(\Gamma,\mathbb{H}) \xrightarrow{B_*} \mathfrak{Z}'(\Gamma,\mathbb{G})$$

is a fibration of 2-groupoids. The long exact sequence of Proposition 11.3 is the fiber homotopy exact sequence associated to this fibration.

The proof of the above proposition is not hard (and one can say it is "standard"), but it is not in the spirit of these notes, so we omit it.

The following is an immediate corollary of Proposition 11.3.

Proposition 11.5. *Let* $\mathbb{G}\colon [\partial\colon G_1 \to G_0]$ *be a* Γ*-crossed-module. Then, we have the exact sequences*

$$1 \longrightarrow H^1(\Gamma, \ker\partial) \longrightarrow H^0(\Gamma, \mathbb{G}) \longrightarrow (\operatorname{coker}\partial)^{\Gamma} \longrightarrow H^2(\Gamma, \ker\partial) \longrightarrow H^1(\Gamma, \mathbb{G}) \longrightarrow H^1(\Gamma, \operatorname{coker}\partial)$$

and

$$1 \longrightarrow H^{-1}(\Gamma, \mathbb{G}) \longrightarrow G_1 \longrightarrow G_0 \longrightarrow H^0(\Gamma, \mathbb{G}) \longrightarrow H^1(\Gamma, G_1) \longrightarrow H^1(\Gamma, G_0) \longrightarrow H^1(\Gamma, \mathbb{G}).$$

Proof. For the first sequence, apply Proposition 11.3 to the short exact sequence of crossed-modules

$$1 \to [\ker\partial \to 1] \to \mathbb{G} \to [1 \to \operatorname{coker}\partial] \to 1.$$

For the second sequence, apply Proposition 11.3 to the short exact sequence

$$1 \to G_1 \to G_0 \to \mathbb{G} \to 1.$$

(To see why these two sequences of crossed-modules are exact use Example 11.1.) □

Remark 11.6. The first exact sequence in Proposition 11.5 can be extended by adding an $H^3(\Gamma, \ker\partial)$ to the right end of it. We do not have the tools to give a systematic proof here but with some effort one can prove it by hand. Also, in the second exact sequence, if G_1 is abelian, the sequence can be extended by $H^2(\Gamma, G_1)$. If G_0 is also abelian, then the sequence can be extended further by $H^2(\Gamma, G_0)$.

Remark 11.7. The inclusion map $\ker\partial \to G_1$ and the projection map $G_0 \to \operatorname{coker}\partial$ induce maps $H^i(\Gamma, \ker\partial) \to H^i(\Gamma, G_1)$ and $H^i(\Gamma, G_0) \to H^i(\Gamma, \operatorname{coker}\partial)$. These maps intertwine the two exact sequences of Proposition 11.5 into a commutative diagram.

12. Appendix: review of 2-crossed-modules and braided crossed-modules

For the convenience of the reader, in this appendix we collect some elementary facts about braided crossed-modules and 2-crossed-modules [BrGi, Con]. We begin with some definitions.

A **crossed-module in groupoids** ([BrGi], page 54) is a morphism of groupoids

$$\mathcal{M} \xrightarrow{\partial} \mathcal{N}$$

such that $\mathcal{M} = \coprod_{x\in \mathrm{Ob}(\mathcal{N})} \mathcal{M}(x)$ is a disjoint union of groups indexed by the set of objects of $\mathcal{N}$. We also have a right action of $\mathcal{N}$ on $\mathcal{M}$ such that an arrow $g \in \mathcal{N}(x,y)$ takes $\alpha \in \mathcal{M}(x)$ to $\alpha^g \in \mathcal{M}(y)$. We require that ∂ satisfy the two axioms of a crossed-module. That is, ∂ is $\mathcal{N}$-equivariant for the right conjugation action of $\mathcal{N}$ on itself, and for every two arrows α, β in $\mathcal{M}$, we have $\alpha^{\partial\beta} = \beta^{-1}\alpha\beta$.

Any crossed-module $[M \to N]$ gives rise to a crossed-module in groupoids

$$\big[M \to [N \rightrightarrows 1]\big].$$

Conversely, to any object x in crossed-module in groupoids $[\partial\colon \mathcal{M} \to \mathcal{N}]$ we can associate a crossed-module $[\partial_x\colon \mathcal{M}(x) \to \mathcal{N}(x)]$ which we call the *automorphism crossed-module* of x. Here, by $\mathcal{N}(x)$ we mean the automorphism group of the object $x \in \mathcal{N}$.

A **2-crossed-module** ([Con], Definition 2.2; also see [BrGi], page 66), is a sequence

$$[L \xrightarrow{\partial} M \xrightarrow{\partial} N]$$

of groups endowed with a right action action of N on M and L, a right action of M on L, and a bracket $\{,\}\colon M \times M \to N$ satisfying the following axioms:

- Let N act on itself by right conjugation. Then both differentials ∂ are G_1-equivariant, and $\partial^2 = 0$;
- For every $g, h \in M$, $\partial\{g,h\} = g^{-1}h^{-1}gh^{\partial g}$;
- For every $g \in M$ and $\alpha \in L$, $\{\partial\alpha, g\} = \alpha^{-1}\alpha^g$ and $\{g, \partial\alpha\} = (\alpha^{-1})^g\alpha^{\partial g}$;
- For every $g, h, k \in M$, $\{g, hk\} = \{g,k\}\{g,h\}^{k^{\partial g}}$;
- For every $g, h, k \in M$, $\{gh, k\} = \{g,k\}^h\{h, k^{\partial g}\}$;
- or every $g, h \in M$ and $x \in N$, $\{g,h\}^x = \{g^x, h^x\}$.

By setting $N = \{1\}$ in the definition of a 2-crossed-module, we obtain the definition of a braided crossed-modules. More precisely, a crossed-module

$$[L \to M]$$

is **braided** if it is endowed with a bracket $\{,\}\colon M \times M \to L$ which satisfies the following axioms:

- For every $g, h \in M$, $\partial\{g,h\} = g^{-1}h^{-1}gh$;
- For every $g \in M$ and $\alpha \in L$, $\{\partial\alpha, g\} = \alpha^{-1}\alpha^g$ and $\{g, \partial\alpha\} = (\alpha^{-1})^g\alpha$;
- For every $g, h, k \in M$, $\{g, hk\} = \{g,k\}\{g,h\}^k$;
- For every $g, h, k \in M$, $\{gh, k\} = \{g,k\}^h\{h,k\}$.

Any 2-crossed-module $[L \to M \to N]$ gives rise to a crossed-module in groupoids

$$\big[\coprod_{x \in N} L(x) \to [N \times M \rightrightarrows N]\big],$$

where $L(x) = L$ and $[N \times M \rightrightarrows N]$ is the action groupoid of the right multiplication action of M on N via ∂. If we view $1 \in N$ as an object in the above crossed-module in groupoids, its automorphism crossed-module is equal to $[L \to \ker\partial]$. This is a braided crossed-module. Conversely, any braided crossed-module $[L \to M]$ gives rise to a 2-crossed-module

$$[L \xrightarrow{\partial} M \xrightarrow{\partial} 1].$$

A braided crossed-module $[L \to M]$ is **symmetric** if for every $g, h \in M$ we have

$$\{g,h\}\{h,g\} = 1.$$

If, in addition, we have

$$\{g,g\} = 1$$

for every $g \in M$, we say that $[L \to M]$ is **Picard**.

The above observation about braided crossed-modules can be used to define a **braided 2-crossed-module** as follows.[2] We say that a 2-crossed-module

$$[K \xrightarrow{\partial} L \xrightarrow{\partial} M]$$

is braided if the sequence

$$K \xrightarrow{\partial} L \xrightarrow{\partial} M \xrightarrow{\partial} 1$$

is endowed with the structure of a 3-crossed-module in the sense of [ArKuUs], Definition 8. That is, we have seven brackets

$$\begin{aligned} \{,\}_{(1)(0)},\ \ \{,\}_{(0)(2)},\ \ \{,\}_{(2)(1)}:&\ \ L\times L\to K\\ \{,\}_{(1,0)(2)},\ \ \{,\}_{(2,0)(1)}:&\ \ M\times L\to K\\ \{,\}_{(0)(2,1)}:&\ \ L\times M\to K\\ \{,\}:&\ \ M\times M\to L \end{aligned}$$

satisfying axioms (**3CM1**)-(**3CM18**) of *loc. cit.*[3] In fact, it follows from the axioms that the two brackets $\{,\}_{(1)(0)}$ and $\{,\}_{(0)(2)}$ are determined by $\{,\}_{(2)(1)}$, and $\{,\}_{(2)(1)}$ itself is the bracket that already comes with the 2-crossed-module. So, to put a braiding on a given 2-crossed-module we have to introduce four new brackets, namely, the last four in the above list. (In our application in §5.3, three of these four brackets are trivial and only $\{,\}: M\times M\to L$ is non-trivial.)

Remark 12.1. It is useful to keep in mind the homotopy theoretic interpretations of the above notions. A crossed-module corresponds to a pointed homotopy 2-type. A crossed-module in groupoids corresponds to an arbitrary homotopy 2-type. A 2-crossed-module corresponds to a pointed homotopy 3-type. Associating a crossed-module in groupoids to a 2-crossed-module corresponds to taking the based loop space. The 2-crossed-module associated to a braided crossed-module corresponds to delooping.

12.1. **Cohomologies of a crossed-module in groupoids.** To be compatible with the rest of the paper, we make the (unusual) assumption that our crossed-module in groupoids $[\partial\colon \mathcal{M}\to\mathcal{N}]$ is sitting in degrees $[-1,1]$. That is, we think of objects of $\mathcal{N}$ as sitting in degree 1, its arrows in degree 0, and arrows of $\mathcal{M}$ in degree -1. We then define H^1 to be the set of connected components of $\mathcal{N}$; this is just a set. For a fixed a base point $x\in \mathrm{Ob}(\mathcal{N})$, we define H^0 and H^{-1} to be, respectively, the cokernel and the kernel of ∂_x in the automorphism crossed-module $[\partial_x\colon \mathcal{M}(x)\to\mathcal{N}(x)]$ of x. Note that H^0 is a group and H^{-1} is an abelian group

In the case where our crossed-module in groupoids comes from a 2-crossed-module $L\to M\to N$, concentrated in degrees $[-1,1]$, the cohomologies defined above are naturally isomorphic to the cohomologies of the 2-crossed-module. In this situation, H^1 is also a group and H^0 is abelian.

12.2. **The 2-groupoid associated to a crossed-module in groupoids.** To any crossed-module in groupoids $[\partial\colon \mathcal{M}\to\mathcal{N}]$ we can associate a strict 2-groupoid $[\mathcal{N}/\mathcal{M}]$ as follows. The objects and the arrows of $[\mathcal{N}/\mathcal{M}]$ are the ones of $\mathcal{N}$. Given two arrows $g,h\in\mathcal{N}(x,y)$, a 2-arrow $g\Rightarrow h$ is an element $\alpha\in\mathcal{M}(y)$ such that $g\partial_y(\alpha)=h$. The composition of two 2-arrows $\alpha\colon g\Rightarrow h$ and $\beta\colon h\Rightarrow k$ is $\alpha\beta$.

[2]To our knowledege, braided 2-crossed-modules were first defined in P. Carrasco's thesis [Car].

[3]We need to modify the axioms of *loc. cit* to account for the fact that our conventions for the actions (left or right) and the brackets, and as a consequenece our 2-crossed-module axioms, are different from those of *loc. cit.*

If $k \in \mathcal{N}(y,z)$, then $\alpha k\colon gk \Rightarrow hk$ is defined to be the 2-arrow corresponding to $\alpha^k \in \mathcal{N}(z)$. If $k \in \mathcal{N}(z,x)$, then $k\alpha\colon kg \Rightarrow kh$ is defined to be the 2-arrow corresponding to α itself.

In the case where $[\partial\colon \mathcal{M} \to \mathcal{N}]$ comes from a 2-crossed-module, the 2-groupoid $[\mathcal{N}/\mathcal{M}]$ can be delooped to a 3-group. That is, there is a (weak) 3-groupoid with one object such that the morphisms from the unique object to itself is equal to $[\mathcal{N}/\mathcal{M}]$. This is true because $[\mathcal{N}/\mathcal{M}]$ is a strict group object in the category of 2-groupoids. (Note that, although this is a strict group object, the multiplication functor is lax. The laxness of the multiplication functor is measured by the bracket of the 2-crossed-module.)

12.3. **Some useful identities.** The following identities are frequently used in the (omitted) proofs of many of the claims in these notes. The proofs are left to the reader. In what follows $[M \to N]$ is a braided crossed-module.

- For every $g,h \in N$, $\{g,h^{-1}\}^h = \{g,h\}^{-1} = \{g^{-1},h\}^g$.
- For every $g,h \in N$, $\{g^{-1},h^{-1}\}^{gh} = \{g,h\}$.
- For every $g \in N$, $\{g,g\}^g = \{g,g\}$.
- For every $g,h,k \in N$, $\{gh,k\} = \{h,g^{-1}kg\}\{g,k\}$.
- For every $g,h,k \in N$. $\{g,hk\} = \{g,h\}\{h^{-1}gh,k\}$.
- For every $g,h,k \in N$, $\{g,h\}^k = \{k^{-1}gk, k^{-1}hk\}$.

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