# SINGULAR CHAINS ON TOPOLOGICAL STACKS, I

THOMAS COYNE, BEHRANG NOOHI

Abstract. We extend the functor Sing of singular chains to the category of topological stacks and establish its main properties. We prove that Sing respects weak equivalences and takes a morphism of topological stacks that is both a Serre and a Reedy fibration to a Kan fibration of simplicial sets. When restricted to the category of topological spaces Sing coincides with the usual singular functor.

Contents



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## 1. Introduction

This is the first instalment of a two-part paper investigating singular chains on topological stacks.

Given a topological stack $\mathcal{X}$ we define the simplicial set $\mathrm{Sing}(\mathcal{X})$ of singular chains on $\mathcal{X}$ and establish its main properties. This generalizes the usual functor $\mathrm{Sing} : \mathsf{Top} \to \mathsf{sSet}$ of singular chains on topological spaces. We address the following questions about $\mathrm{Sing}(\mathcal{X})$: functoriality with respect to morphisms of stacks, the homotopy type of $\mathrm{Sing}(\mathcal{X})$, and the effect on fibrations of topological stacks.

**Functoriality and the homotopy type of** $\mathrm{Sing}(\mathcal{X})$**.** There are several ways to define the homotopy type of a topological stack (see for instance [Be, Ha, Mo, No12]). In [No12] the notion of *classifying space* of a topological stack is introduced to give a better grip on the functoriality of the homotopy type. Nevertheless, the functoriality of the classifying space only makes sense in the homotopy category of topological spaces, that is, the classifying space is a functor $\mathrm{CS} : \mathsf{topStack} \to \mathrm{Ho}(\mathsf{Top})$.

Our construction of singular chains in this paper enhances this by giving us an honest functor $\mathrm{Sing} : \mathsf{topStack} \to \mathsf{sSet}$. When restricted to the subcategory $\mathsf{Top}$, this functor coincides with the usual singular functor on topological spaces. The functor $\mathrm{Sing} : \mathsf{topStack} \to \mathsf{sSet}$ in fact lifts the classifying space functor CS,

$$\begin{array}{ccc} \mathsf{topStack} & \xrightarrow{\mathrm{Sing}} & \mathsf{sSet} \\ {\scriptstyle \mathrm{CS}}\downarrow & & \downarrow \\ \mathrm{Ho}(\mathsf{Top}) & \xrightarrow[\sim]{} & \mathrm{Ho}(\mathsf{sSet}) \end{array}$$

at least for the full subcategory of $\mathsf{topStack}$ consisting of Serre stacks. This is a consequence of one of our main results (Theorem 12.2).

**Theorem 1.1.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a weak equivalence of Serre stacks. Then,* $\mathrm{Sing}(f) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ *is a weak equivalence of simplicial sets. In particular, if $X \to \mathcal{X}$ is a classifying space for $\mathcal{X}$, then the induced map* $\mathrm{Sing}(X) \to \mathrm{Sing}(\mathcal{X})$ *is a weak equivalence.*

In particular, $\mathrm{Sing}(\mathcal{X})$ has the same homotopy type as the classifying space $\mathrm{CS}(\mathcal{X})$. Somewhat surprisingly, the proof of the above theorem is highly nontrivial.

**Effect on fibrations of topological stacks.** It is well known that for a Serre fibration $f : X \to Y$ of topological spaces, the induced map $\mathrm{Sing}(f) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a Kan fibration of simplicial sets. The corresponding statement for topological stacks, however, should be formulated more carefully, as there are various notions of fibrations between topological stacks. For example, the above statement would clearly be false if we use the notion of Serre fibration for topological stack as in ([No14], Definition 3.6), because this notion is "intrinsic" (i.e., is invariant under replacing a stack by an equivalent stack – in particular, any equivalence of topological stacks $f : \mathcal{X} \to \mathcal{Y}$, such as the inclusion of a point into a trivial groupoid, is automatically a Serre fibration).

It turns out, the correct condition on a morphism $f : \mathcal{X} \to \mathcal{Y}$ to ensure that $\mathrm{Sing}(f) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ is a Kan fibration is that $f$ is a Serre fibration and also a Reedy fibration (Definition 8.10). This is another main result of the paper (Theorem 11.8).

**Theorem 1.2.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks that is a (weak) Serre fibration and also a Reedy fibration. Then,* $\mathrm{Sing}(p) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ *is a (weak) Kan fibration.*

We point out that the Reedy condition can always be arranged for any morphism of stacks: given $f : \mathcal{X} \to \mathcal{Y}$ we can replace $\mathcal{X}$ by an equivalent stack $\mathcal{X}'$ such that the corresponding morphism $f' : \mathcal{X}' \to \mathcal{Y}$ is a Reedy fibration (Proposition 8.14). Such a replacement would not affect the property of being a Serre fibration.

The paper is organized as follows. In Section 3, we set up the terminology and review some generalities about stacks and topological stacks. In Section 4, we introduce the *tilde* construction. This is the left Kan extension along the inclusion $\mathbf{\Delta} \to \mathsf{Top}$ and plays a crucial role in the rest of the paper. In Sections 6-7 we review some basic facts about homotopy of maps between morphisms of stacks. We also recall the relevant background on fibrations of stacks. The only new notion in this section is that of a *restricted* homotopy (6.2) which is related to the tilde construction introduced in Section 4.

In Section 8 we look at various model structures on the categories of groupoids, presheaves of groupoids and simplicial groupoids, and establish some of their properties which are, presumably, well known but which we have been unable to locate in the literature. The notion of Reedy fibration of stacks (Definition 8.10) introduced and studied in this section is central to the paper. It is an adaptation of Reedy fibration of simplicial groupoids.

We introduce the functor $\mathrm{Sing} : \mathsf{topStack} \to \mathsf{sSet}$ in Section 9. Section 10 is the technical heart of the paper where we prove a list of lemmas which play key role in the proofs of our main results. In Section 11 we prove the first main result of the paper, namely, that if $f : \mathcal{X} \to \mathcal{Y}$ is both a Serre and a Reedy fibration, then $\mathrm{Sing}(f)$ is a Kan fibration (Theorem 11.8).

In Section 12 we use the results of Section 11 to prove the second main result of the paper, namely, that Sing preserves weak equivalences (Theorem 12.2). In particular, this implies that the (singular simplicial set of the) classifying space of a topological stack $\mathcal{X}$ is naturally weakly homotopy equivalent to $\mathrm{Sing}(\mathcal{X})$, see Proposition 12.1.

In the subsequent paper on the subject we study the adjunction between Sing and geometric realization, as well as the effect of the functor Sing on the totalization of cosimplicial stacks. We use these results to study singular chains on mapping stacks.

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## 2. Notation and terminology

2.1. **Stacks and Yoneda.** We will use the rather unconventional approach of working with *presheaves of groupoids* rather than *categories fibered in groupoids*. We use calligraphic symbols $\mathcal{X}$, $\mathcal{Y}$, etc. for presheaves of groupoids.

We often regard a topological space $X$ as a stack via Yoneda embedding. We use the same notation $X$ for the functor represented by $X$.

2.2. **Strict versus 2-categorical limits.** When we talk about (co)limits in a 2-category $\mathsf{C}$ we always mean the *strict* ones. Otherwise, we call them 2-categorical (co)limits, or 2-(co)limits.

In particular, for (presheaves of) groupoids $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, we denote their

$$\text{strict fiber product by } \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$$

and their

$$\text{2-fiber product by } \mathcal{X} \tilde{\times}_{\mathcal{Y}} \mathcal{Z}.$$

The notation $\mathcal{X} \cong \mathcal{Y}$ means an isomorphism of (presheaves of) groupoids, and $\mathcal{X} \sim \mathcal{Y}$ means an equivalence of (presheaves of) groupoids.

2.3. **Composition of morphisms in categories.** We use functional notation $g \circ f$ for composition of 1-morphisms $f : X \to Y$ and $g : Y \to Z$, and multiplicative notation $\alpha \cdot \beta$ (or simply $\alpha\beta$) for composition of 2-isomorphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$. We use the notation $h \circ \alpha$ for the composition of a 2-isomorphism $\alpha : f \Rightarrow g$ between $f, g : X \to Y$ with a morphism $h : Y \to Z$.

2.4. **Categories of interest.** We usually use the notation $[\mathsf{C}, \mathsf{D}]$ for functor categories. We will be working with the following categories:

- $\mathsf{Top}$, the category of compactly generated Hausdorff spaces;
- $\mathsf{Gpd}$, the category of small groupoids;
- $\mathsf{pshSet}$, the category of presheaves of sets over $\mathsf{Top}$;
- $\mathsf{pshGpd}$, the category of presheaves of groupoids over $\mathsf{Top}$;
- $\mathsf{sSet}$, the category of simplicial sets;
- $\mathsf{sGpd}$, the category of simplicial groupoids;
- $\mathsf{bsSet}$, the category of bisimplicial sets.

Note that $\mathsf{Gpd}$, $\mathsf{pshGpd}$ and $\mathsf{sGpd}$ carry a 2-category structure; we will use the same notation for the corresponding 2-categories.

2.5. **Simplicial sets.** The category of finite ordinal numbers with order preserving maps between them is denoted by $\mathbf{\Delta}$. The simplicial $n$-simplex is denoted by $\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$. The topological $n$-simplex is denoted by

$$|\Delta^n| = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$$

We denote the cosimplicial object $n \mapsto |\Delta^n|$ in $\mathsf{Top}$ by $|\Delta^\bullet|$. The $k^{\text{th}}$ horn in $\Delta^n$, namely, the sub-simplicial set of $\Delta^n$ generated by the $i^{\text{th}}$ faces of the unique non-degenerate $n$-cell in $\Delta^n$, $i \in \{0, 1, \cdots, \hat{k}, \cdots, n\}$, is denoted by $\Lambda^n_k$. When talking about homotopies between maps we often use the notation $[0,1]$ instead of $|\Delta^1|$.

The bisimplex $\Delta^{m,n}$ is the bisimplicial set $\Delta^{m,n} : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$ represented by $([m],[n]) \in \mathbf{\Delta} \times \mathbf{\Delta}$. That is, $\Delta^{m,n} := \mathrm{Hom}_{\mathbf{\Delta}\times\mathbf{\Delta}}(-, ([m],[n])) = \Delta^m \boxtimes \Delta^n$ (see Section 11.3).

For a simplicial set $X \in \mathsf{sSet}$, we use the notation $\tilde{X} \in \mathsf{pshGpd}$ for the left Kan extension of $X$ along $\mathbf{\Delta} \to \mathsf{Top}$ (more details can be found in Section 4).

## 3. Topological stacks

Throughout the paper, we will work over the base Grothendieck site $\mathsf{Top}$ of compactly generated Hausdorff topological spaces (with the open-cover topology). We will use the rather unconventional approach of working with *presheaves of groupoids* rather than *categories fibered in groupoids* (there is a natural strictification functor from the latter to the former). The equivalence of this approach with Grothendieck's approach via fibered categories has been worked out in [Ho] (also see Section 3.3 below).

3.1. **Presheaves of groupoids.** We denote the 2-category of presheaves of groupoids over $\mathsf{Top}$ by $\mathsf{pshGpd} = [\mathsf{Top}^{\mathrm{op}}, \mathsf{Gpd}]$. We shall denote the objects of this category with calligraphic letters, i.e., $\mathcal{X} \in \mathsf{pshGpd}$. For $T \in \mathsf{Top}$ we call $\mathcal{X}(T)$ the *groupoid of $T$-points* of $\mathcal{X}$.

By an *equivalence* of presheaves of groupoids we mean a morphism $f : \mathcal{X} \to \mathcal{Y}$ such that for every $T \in \mathsf{Top}$, the induced map $f(T) : \mathcal{X}(T) \to \mathcal{Y}(T)$ on the $T$-points is an equivalence of groupoids. Two presheaves of groupoids are *equivalent* if there exists a zigzag of equivalences between them.

3.2. **Yoneda.** Let $\mathsf{pshSet} = [\mathsf{Top}^{\mathrm{op}}, \mathsf{Set}]$ be the category of presheaves of sets over the category $\mathsf{Top}$ of topological spaces. Regarding a set as a a groupoid in which the only morphisms are the identity morphisms, we identify $\mathsf{pshSet}$ with a full subcategory $\mathsf{pshGpd}$ of the category of presheaves of groupoids over the category $\mathsf{Top}$.

The Yoneda functor $\mathsf{Top} \to \mathsf{pshSet}$ (or $\mathsf{Top} \to \mathsf{pshGpd}$) sends a topological space $X$ to the functor $\mathrm{Hom}_{\mathsf{Top}}(-, X)$ represented by $X$. This identifies $\mathsf{Top}$ with a full subcategory $\mathsf{pshSet}$ (or $\mathsf{pshGpd}$). More precisely, we have the following.

**Lemma 3.1.** *Let $X$ be a topological space and $\mathcal{Y}$ a presheaf of groupoids. Then, there is a natural isomorphism of groupoids* $\mathrm{Hom}_{\mathsf{pshGpd}}(X, \mathcal{Y}) \cong \mathcal{Y}(X)$.

As in the above lemma, we often abuse notation and use the same notation both for $X \in \mathsf{Top}$ and for the image of $X$ in $\mathsf{pshSet}$ (or $\mathsf{pshGpd}$) under the Yoneda functor.

The Yoneda embedding preserves fiber products (in fact, all limits), but it seldom preserves colimits. If a presheaf $\mathcal{X}$ is equivalent to $\mathrm{Hom}_{\mathsf{Top}}(-, X)$, for some $X \in \mathsf{Top}$, we often abuse terminology and say that $\mathcal{X}$ is a topological space.

3.3. **Stacks.** Following ([Ho], Definition 1.3) we define a *stack* over $\mathsf{Top}$ to be a presheaf of groupoids $\mathcal{X} \in \mathsf{pshGpd}$ that satisfies the descent condition

$$\mathcal{X}(T) \xrightarrow{\sim} \mathrm{holim}\Big( \prod \mathcal{X}(U_i) \rightrightarrows \prod \mathcal{X}(U_{ij}) \Rrightarrow \prod \mathcal{X}(U_{ijk}) \Big)$$

for every $T \in \mathsf{Top}$ and every open cover $\{U_i\}$ of $T$. Morphisms and 2-isomorphisms of stacks are the ones of the underlying presheaves of groupoids. That is, stacks form a full sub-2-category of $\mathsf{pshGpd}$.

As shown in [Ho], the presheaf approach to stacks is equivalent to the approach via categories fibered in groupoids. Let us elaborate on this. The projective model structure on (the 1-category underlying) $\mathsf{pshGpd}$ is Quillen equivalent to the projective model structure (in the sense of [Ho], Theorem 4.2) on (the 1-category of) categories fibered in groupoids over $\mathsf{Top}$; see [Ho], Corollary 4.3. The underlying Quillen adjunction is defined as follows: to any category $\mathcal{C}$ fibered in groupoids over $\mathsf{Top}$ we associate the presheaf of groupoids

$$T \in \mathsf{Top}, \quad T \mapsto \mathrm{Hom}_{\mathrm{FibCat}}(T, \mathcal{C}) \ \in \mathsf{Gpd}.$$

The left adjoint to this functor is given by the Grothendieck construction.

This Quillen equivalence gives rise to a Quillen equivalence between the localizations of both model categories with respect to hypercovers; see [Ho], Corollary 4.5. The fibrant objects in either of these localized model categories are called stacks.

3.4. **Topological stacks.** By a *topological stack* we mean a stack over $\mathsf{Top}$ which is equivalent to the quotient stack of a topological groupoid $\mathbb{X} = [R \rightrightarrows X]$, with $R$ and $X$ topological spaces. A topological stack is *Serre* if it has a groupoid presentation such that $s : R \to X$ is locally (on source and target) a Serre fibration. That is, for every $y \in R$, $s$ is a Serre fibration from a neighborhood of $y$ to a neighborhood of $s(y)$.

Morphisms and 2-isomorphisms of topological stacks are the ones of the underlying presheaves of groupoids, so topological stacks, as well as Serre topological stacks, form a full sub-2-category of $\mathsf{pshGpd}$. We denote the 2-category of topological stacks by $\mathsf{topStack}$.

3.5. **Strict and 2-categorical fiber products.** Consider the following diagram in the 2-category $\mathsf{Gpd}$ of groupoids:

$$\begin{array}{ccc} & & K \\ & & \downarrow{\scriptstyle \mathrm{p}} \\ H & \xrightarrow[\mathrm{q}]{} & G \end{array}$$

Recall that the *2-fiber product* (or *2-categorical fiber product*)

$$H \tilde{\times}_G K$$

has objects triples $(x, y, \varphi)$, where $x$ is an object in $H$, $y$ is an object in $K$ and $\varphi : q(x) \to p(y)$ is a morphism in $G$. A morphism from $(x, y, \varphi)$ to $(x', y', \varphi')$ is a pair of morphisms $\alpha : x \to x'$ and $\beta : y \to y'$, in $H$ and $K$ respectively, such that $\varphi' \circ q(\alpha) = p(\beta) \circ \varphi$.

There is a fully faithful functor

$$H \times_G K \to H \tilde{\times}_G K$$

from the strict fiber product to the 2-fiber product, sending a pair $(x, y) \in H \times_G K$ to the triple $(x, y, \mathrm{id})$. The image consists of those triples $(x, y, \varphi)$ with $\varphi = \mathrm{id}$. This map is sometimes an equivalence (Lemma 8.2) but not always.

The strict and 2-categorical product are defined objectwise for presheaves of groupoids, namely

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})(T) = \mathcal{X}(T) \times_{\mathcal{Z}(T)} \mathcal{Y}(T), \quad \forall T \in \mathsf{Top}$$

and

$$(\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y})(T) := \mathcal{X}(T) \tilde{\times}_{\mathcal{Z}(T)} \mathcal{Y}(T), \quad \forall T \in \mathsf{Top}.$$

**Lemma 3.2.** *The 2-categories of stacks, topological stacks and Serre topological stacks are all closed under 2-fiber products (in fact, all finite 2-limits), and these are computed as presheaves of groupoids.*

*Proof.* In the case of stacks this is well known (homotopy limit commutes with 2-fiber product). For the other two cases see ([No05], page 30) for the construction of a groupoid presentation for $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ out of those for $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. □

*Remark* 3.3. The reason for using the nonstandard notation $\tilde{\times}$ is that in this paper we will mostly be using *strict* fiber products of (presheaves of) groupoids and we need to distinguish between the two notions.

3.6. **Classifying spaces for topological stacks.** The following theorem has been proven in ([No14], Corollary 3.17).

**Theorem 3.4.** *Let $\mathcal{X}$ be a topological stack. Then, there exists an atlas $\varphi : X \to \mathcal{X}$ that is a trivial weak Serre fibration. This means that, for any map from a topological space $T$, the fiber product*

$$\begin{array}{ccc} X \tilde{\times}_{\mathcal{X}} T & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \xrightarrow[\varphi]{} & \mathcal{X} \end{array}$$

*has the property that $X \tilde{\times}_{\mathcal{X}} T \to T$ is a trivial weak Serre fibration of topological spaces (in particular, a weak homotopy equivalence).*

See Definition 7.3 for the general definition of trivial weak Serre fibration, bearing in mind that the definition simplifies considerably in the case of topological spaces.

We call a map $\varphi : X \to \mathcal{X}$ as in Theorem 3.4 a *classifying atlas* for $\mathcal{X}$. Note that in the definition of classifying atlas given in [No12] we only require $\varphi : X \to \mathcal{X}$ to be a universal weak equivalence. The definition we are using here is stronger.

The $n^{\text{th}}$ *homotopy group* (set if $n = 0$) of a pointed topological stack $(\mathcal{X}, x)$ is defined ([No14], Section 5) to be the group $\pi_n(\mathcal{X}, x) = [(S^n, s_0), (\mathcal{X}, x)]$ of homotopy classes of pointed maps. Equivalently, it can be defined to be the homotopy group $\pi_n(X, x')$ of a classifying atlas $X$ for $\mathcal{X}$ at some lift $x'$ of $x$ to $X$. This definition is independent of the choice of $X$ and $x'$ (up to a natural isomorphism).

A morphism of topological stacks $f : \mathcal{X} \to \mathcal{Y}$ is called a *weak equivalence* if it induces isomorphisms $f_* : \pi_n(\mathcal{X}, x) \to \pi_n(\mathcal{Y}, y)$ for all choices of basepoint and all $n \geq 0$.

## 4. The tilde construction

Consider the inclusion $\mathbf{\Delta} \to \mathsf{Top}$, $[n] \mapsto |\Delta^n|$. Left Kan extension along this inclusion gives rise to a functor

$$\begin{aligned} \mathsf{sSet} &\to \mathsf{pshSet} \quad (\hookrightarrow \mathsf{pshGpd}) \\ A &\mapsto \tilde{A} \end{aligned}$$

which is uniquely determined by the property that it preserves colimits and sends $\Delta^n$ to $|\Delta^n|$ (rather, the presheaf represented by it). It is left adjoint to the restriction functor

$$\begin{aligned} -_{\mathbf{\Delta}} : \mathsf{pshSet} &\to \mathsf{sSet} \quad (\hookrightarrow \mathsf{sGpd}) \\ X &\mapsto X_{\mathbf{\Delta}} = \mathrm{Hom}_{\mathsf{pshSet}}(|\Delta^\bullet|, X). \end{aligned}$$

More explicitly, $\tilde{A}$ is constructed exactly like the colimit construction of the geometric realization of $A$, except that instead of using the topological simplices $|\Delta^n|$ as building blocks we use the presheaves in $\mathsf{pshSet}$ represented by them.

We have a natural map

$$\psi_A : \tilde{A} \to |A|. \tag{4.1}$$

This is adjoint to the map $A \to \mathrm{Sing}(|A|) = |A|_{\mathbf{\Delta}}$, the unit of the adjunction $| - | : \mathsf{Top} \rightleftharpoons \mathsf{sSet} : \mathrm{Sing}$. Note that the Yoneda embedding $\mathsf{Top} \to \mathsf{pshSet}$ (or $\mathsf{pshGpd}$) does not necessarily preserve colimits, so $\psi_A$ is often not an isomorphism (but it is when $A = \Delta^n$).

*Remark* 4.1. The standard notation in the literature for the restriction functor along $\mathbf{\Delta} \to \mathsf{Top}$ (rather, along $i : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Top}^{\mathrm{op}}$) and its left adjoint, the left Kan extension, are $i^*$ and $i_!$, respectively. Our choice of the alternative notation $()_\Delta$ and $()^\sim$ is only to reduce the burden of notation and enhance readability of the long formulas we will encounter.

*Example* 4.2. Write $\Lambda^n_k$ as the coequalizer of

$$\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{i \in \{0,1,\ldots,n\}, i \neq k} \Delta^{n-1} \to \Lambda^n_k$$

Then, we can write $\tilde{\Lambda}^n_k$ as the coequalizer

$$\coprod_{0 \leq i < j \leq n} |\Delta^{n-2}| \rightrightarrows \coprod_{i \in \{0,1,\ldots,n\}, i \neq k} |\Delta^{n-1}|$$

in $\mathsf{pshSet}$. The map $\psi_{\Lambda^n_k} : \tilde{\Lambda}^n_k \to |\Lambda^n_k|$ is almost never an isomorphism.

We can extend the restriction functor $-_{\mathbf{\Delta}}$ defined above to $\mathsf{pshGpd}$:

$$\begin{aligned} -_{\mathbf{\Delta}} : \mathsf{pshGpd} &\to \mathsf{sGpd}, \\ \mathcal{X} &\mapsto \mathcal{X}_{\mathbf{\Delta}} = \mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^\bullet|, \mathcal{X}). \end{aligned}$$

We have the following lemma.

**Lemma 4.3.** *Let $A$ be a simplicial set and $\mathcal{X}$ a presheaf of groupoids. Then, we have an isomorphism (and not just an equivalence) of groupoids*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}),$$
$$f \mapsto f_{\mathbf{\Delta}} \circ \iota_A.$$

*Here, $\iota_A : A \to \tilde{A}_{\mathbf{\Delta}}$ is the unit of adjunction. In particular, we have the following natural isomorphisms*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{pshGpd}}(|\Delta^n|, \mathcal{X}) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, \mathcal{X}_{\mathbf{\Delta}}) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & \mathcal{X}(|\Delta^n|) & \end{array}$$

*Proof.* In the case where $\mathcal{X}$ is a presheaf of sets, i.e., $\mathcal{X} \in \mathsf{pshSet}$, this is just the left adjointness of the left Kan extension. For the general case view $\mathcal{X}$ as a groupoid object in $\mathsf{pshSet}$ and apply the above isomorphisms to $\mathrm{Ob}(\mathcal{X})$ and $\mathrm{Mor}(\mathcal{X}) \in \mathsf{pshSet}$. □

## 5. Yoneda and colimits

As we pointed out in the previous section, unless $A$ is representable, the natural map $\psi_A : \tilde{A} \to |A|$ is not in general an isomorphism of presheaves of sets. This is due to the fact that the Yoneda functor $\mathsf{Top} \to \mathsf{pshSet}$ (or $\mathsf{Top} \to \mathsf{pshGpd}$) does not preserve colimits.

In certain situations, however, we have the following partial result.

**Lemma 5.1.** *Let $\mathcal{X}$ be a Serre topological stack. Let $A \hookrightarrow B$ and $A \hookrightarrow C$ be closed embeddings of topological spaces. Assume both maps are locally trivial Serre cofibrations. Then, the map*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B \coprod_A C, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(B {\coprod_A}' C, \mathcal{X})$$

*induced by the natural map $B \coprod'_A C \to B \coprod_A C$ is an equivalence of groupoids. Here, $\coprod$ stands for colimit in $\mathsf{Top}$ and $\coprod'$ stands for colimit in $\mathsf{pshSet}$ (which is the same as colimit in $\mathsf{pshGpd}$).*

*Proof.* This is an easy consequence of ([BeGiNoXu], Proposition 1.3). Note that ([BeGiNoXu], Proposition 1.3) is proved for Hurewicz stacks. The proof for the case of Serre topological stacks is entirely similar; see ([No05], Proposition 16.1 and Theorem 16.2) for more details.

To prove the lemma, note that the groupoid

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B {\coprod_A}' C, \mathcal{X}) \cong \mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$

can be identified with the full subgroupoid of the groupoid

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$

consisting of those triples $(f, g, \varphi)$,

$$f : B \to \mathcal{X}, \ g : G \to \mathcal{X}, \ \varphi : f|_A \Rightarrow g|_A,$$

for which $f|_A = g|_A$ and $\varphi = \mathrm{id}$. The composition

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B \coprod_A C, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(B {\coprod_A}' C, \mathcal{X})$$
$$\hookrightarrow \mathrm{Hom}_{\mathsf{pshGpd}}(B, \mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C, \mathcal{X})$$

is an equivalence of groupoids by (the Serre version) of ([BeGiNoXu], Proposition 1.3). Since the second functor is fully faithful, it follows that both functors are equivalences of groupoids. □

*Remark* 5.2. In the course of the proof of the above lemma we have also shown that the natural map

$$\mathrm{Hom}_{\mathsf{pshGpd}}(B,\mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C,\mathcal{X}) \hookrightarrow \mathrm{Hom}_{\mathsf{pshGpd}}(B,\mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(A,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(C,\mathcal{X})$$

is an equivalence of groupoids. In other words, the strict and the 2-fiber product are equivalent.

**Definition 5.3.** We say that a simplicial set $A$ has the *gluing property* with respect to a presheaf of groupoids $\mathcal{X}$ if the map

$$\begin{aligned} \Sigma_{A,\mathcal{X}} : \mathrm{Hom}_{\mathsf{pshGpd}}(|A|,\mathcal{X}) &\to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X}) \\ f &\mapsto f \circ \psi_A \end{aligned}$$

is an equivalence of groupoids.

**Lemma 5.4.** *The simplicial $n$-simplex $\Delta^n$ has the gluing property with respect to any presheaf of groupoids $\mathcal{X}$.*

*Proof.* This follows from the fact that $\psi_A : |A| \to \tilde{A}$ is an isomorphism when $A = \Delta^n$. In fact, in this case the maps $\Sigma_{\Delta^n,\mathcal{X}}$ are isomorphisms of groupoids. □

**Lemma 5.5.** *Let $A \hookrightarrow B$ and $A \hookrightarrow C$ be monomorphisms of simplicial sets. If $A$, $B$ and $C$ have the gluing property with respect to a Serre topological stack $\mathcal{X}$, then so does $B \coprod_A C$.*

*Proof.* We have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{pshGpd}}(|B\textstyle\coprod_A C|,\mathcal{X}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|\textstyle\coprod_{|A|}|C|,\mathcal{X}) \\
\text{(Lemma 5.1)} \quad &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|\textstyle\coprod'_{|A|}|C|,\mathcal{X}) \\
\text{(Definition of colimit)} \quad &\xrightarrow{\cong} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|,\mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(|A|,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(|C|,\mathcal{X}) \\
\text{(Remark 5.2)} \quad &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(|B|,\mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(|A|,\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(|C|,\mathcal{X}) \\
\text{(Assumption)} \quad &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B},\mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{C},\mathcal{X})
\end{aligned}$$

Notice that the above equivalence is equal to the following composition:

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{pshGpd}}(|B\textstyle\coprod_A C|,\mathcal{X}) &\xrightarrow{\Sigma_{B\coprod_A C,\mathcal{X}}} \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{B\textstyle\coprod_A C},\mathcal{X}) \cong \\
\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}\textstyle\coprod_{\tilde{A}}\tilde{C},\mathcal{X}) &\cong \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B},\mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{C},\mathcal{X}) \\
&\hookrightarrow \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B},\mathcal{X}) \tilde{\times}_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{C},\mathcal{X}).
\end{aligned}$$

Since the last functor is fully faithful and the composition is shown above to be an equivalence, it follows that $\Sigma_{B\coprod_A C,\mathcal{X}}$ is also an equivalence. □

Recall that a simplicial set $X$ is called *non-singular* ([JaRoWa], Definition 1.2.2) if for every non-degenerate $n$-simplex $x$, the corresponding map $\bar{x} : \Delta^n \to X$ is a monomorphism. Examples we will encounter include A=$\partial\Delta^n$, $\Lambda^n_k$ and $\Lambda^n_k \times \Delta^1$. Non-singular simplicial sets are closed under taking sub-objects and products.

**Corollary 5.6.** *Let $D$ be a finite non-singular simplicial set. Then, $D$ has the gluing property with respect to every Serre topological stack $\mathcal{X}$. That is, for every Serre topological stack $\mathcal{X}$, the map $\psi_D : \tilde{D} \to |D|$ induces an equivalence of groupoids*

$$\Sigma_{D,\mathcal{X}} : \mathrm{Hom}_{\mathsf{pshGpd}}(|D|, \mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{D}, \mathcal{X})$$
$$f \mapsto f \circ \psi_D.$$

*Proof.* Proof proceeds by induction on the total number of non-degenerate simplices of $D$. Choose a maximal non-degenerate simplex $x$, and write $B \subset D$ for the sub simplicial set of $D$ generated by the rest of the non-degenerate simplices. Set $A := B \cap \bar{x}(\Delta^n)$, where $\bar{x} : \Delta^n \to D$ is the map corresponding to $x$; note that this map is a monomorphism by assumption. Also, note that the sets of non-degenerate simplices of $A$ and $B$ are both properly contained in the set of non-degenerate simplices of $D$, so they have a smaller size. By the induction hypothesis, the claim is true for $A$ and $B$, and by Lemma 5.4 it is also true for $\Delta^n$. Therefore, by Lemma 5.5, the claim is true for $D = B \coprod_A \Delta^n$. □

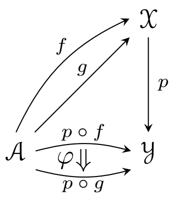
As we pointed out above, in the case $D = \Delta^n$ the above equivalence is indeed an isomorphism of groupoids.

## 6. Homotopy between morphisms of presheaves of groupoids

We review the notion of homotopy between morphisms of stacks from [No14], and introduce a variant called restricted homotopy.

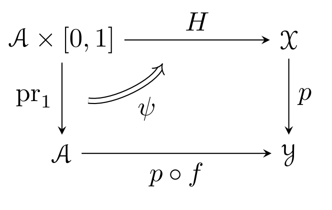
### 6.1. **Fiberwise homotopy.**

**Definition 6.1.** Let $f, g : \mathcal{A} \to \mathcal{X}$ and $p : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids, and $\varphi : p \circ f \Rightarrow p \circ g$ a 2-isomorphism:




A *fiberwise homotopy from $f$ to $g$ relative to $\varphi$* is a quadruple $(H, \epsilon_0, \epsilon_1, \psi)$ where

- $H : \mathcal{A} \times [0,1] \to \mathcal{X}$ is a morphism of presheaves of groupoids;
- $\epsilon_0 : f \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow g$ are 2-isomorphisms;
- $\psi : p \circ f \circ \mathrm{pr}_1 \Rightarrow p \circ H$ is a 2-isomorphism,




such that $\psi_0 = p \circ \epsilon_0$ and $\psi_1 \cdot (p \circ \epsilon_1) = \varphi$.

(Notation: $H_i := H|_{\mathcal{A} \times \{i\}}$, $\psi_i := \psi|_{\mathcal{A} \times \{i\}}$, for $i = 0, 1$.) In the case where $\varphi$ and $\psi$ are both identity 2-isomorphisms (so $p \circ f = p \circ g$ and $p \circ f \circ \mathrm{pr}_1 = p \circ H$) we say that $H$ is a *homotopy relative to* $\mathcal{Y}$.

A fiberwise homotopy as above is called *strict* if $\epsilon_0$ and $\epsilon_1$ are the identity 2-isomorphisms.

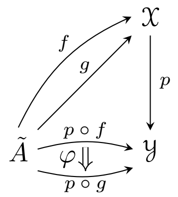
A *ghost fiberwise homotopy from $f$ to $g$ relative to $\varphi$* is a 2-isomorphism $\xi : f \Rightarrow g$ such that $\varphi = p \circ \xi$.

Ghost homotopies typically arise from those quadruples $(H, \epsilon_0, \epsilon_1, \psi)$ for which $H$ and $\psi$ remain constant along $[0, 1]$, that is, they factor through $\mathrm{pr}_1$. In this case, $\xi := \epsilon_0 \cdot \epsilon_1$ is a ghost fiberwise homotopy from $f$ to $g$ relative to $\varphi$. Conversely, from a ghost homotopy $\varphi$ we can construct quadruples $(g \circ \mathrm{pr}_1, \xi, \mathrm{id}, \varphi \circ \mathrm{pr}_1)$ and $(f \circ \mathrm{pr}_1, \mathrm{id}, \xi, \mathrm{id} \circ \mathrm{pr}_1)$.

*Remark* 6.2. There is some flexibility in choosing $H$. More precisely, if $H' : \mathcal{A} \times [0, 1] \to \mathcal{X}$ is 2-isomorphic to $H$ via $\alpha : H \Rightarrow H'$, then $(H', \epsilon'_0, \epsilon'_1, \psi')$ is also a fiberwise homotopy from $f$ to $g$ relative to $\varphi$, where $\epsilon'_0 = \epsilon_0 \cdot \alpha_0$, $\epsilon'_1 = \alpha_1^{-1} \cdot \epsilon_1$ and $\psi' = \psi \cdot (p \circ \alpha)$.

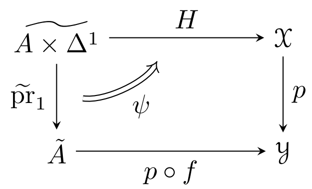
6.2. **Restricted fiberwise homotopy.** The notion of restricted homotopy we introduce below only applies to morphisms of the form $\tilde{A} \to \mathcal{X}$, where $A$ is a simplicial set and $\mathcal{X}$ is a presheaf of groupoids.

**Definition 6.3.** Let $A$ be a simplicial set. Let $f, g : \tilde{A} \to \mathcal{X}$ and $p : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids, and $\varphi : p \circ f \Rightarrow p \circ g$ a 2-isomorphism:



A *restricted fiberwise homotopy from $f$ to $g$ relative to $\varphi$* is a quadruple $(H, \epsilon_0, \epsilon_1, \psi)$ where

- $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ is a morphism of presheaves of groupoids;
- $\epsilon_0 : f \Rightarrow H_0$ and $\epsilon_1 : H_1 \Rightarrow g$ are 2-isomorphisms;
- $\psi : p \circ f \circ \widetilde{\mathrm{pr}}_1 \Rightarrow p \circ H$ is a 2-isomorphism,

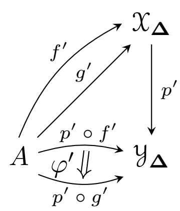


such that $\psi_0 = p \circ \epsilon_0$ and $\psi_1 \cdot (p \circ \epsilon_1) = \varphi$.

(Notation: $H_0 := H \circ \tilde{i}$, where $i : A \to A \times \Delta^1$ is the time 0 map.) In the case where $\varphi$ and $\psi$ are both identity 2-isomorphisms (so $p \circ f = p \circ g$ and $p \circ f \circ \widetilde{\mathrm{pr}}_1 = p \circ H$) we say that $H$ is a *restricted homotopy relative to* $\mathcal{Y}$.

A restricted fiberwise homotopy as above is called *strict* if $\epsilon_0$ and $\epsilon_1$ are the identity 2-isomorphisms.

*Remark* 6.4. In view of the adjunction of Lemma 4.3, we can replace the diagrams above with their corresponding diagram in the category of simplicial groupoids. For example,



Thus, we can regard a restricted homotopy as a homotopy in the category of simplicial groupoids.

*Remark* 6.5. As in Remark 6.2, there is some flexibility in choosing $H$, namely, we are allowed to replace $H$ by any map 2-isomorphic to it (and adjust $\epsilon_0$, $\epsilon_1$ and $\psi$ accordingly).

An ordinary homotopy gives rise to a restricted homotopy.

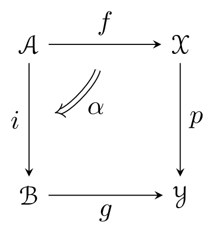
**Lemma 6.6.** *Let $A$ be a simplicial set and let $\mathcal{A} := \tilde{A}$. Notation being as in Definition 6.1, suppose that we are given a fiberwise homotopy $(H, \epsilon_0, \epsilon_1, \psi)$ from $f$ to $g$ relative to $\varphi$. Then, precomposing with the natural map $\widetilde{A \times \Delta^1} \to \tilde{A} \times [0,1]$ gives rise to a restricted fiberwise homotopy from $f$ to $g$ relative to $\varphi$.*

*Proof.* Straightforward. □

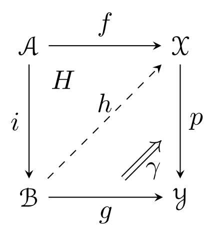
## 7. Lifting conditions

We shall review some of the material from [No14] and recall the notion of (weak) Serre fibration between stacks. For a full account see Sections 2 and 3 of [No14]. Before we start, it is worthwhile to emphasize the difference between the notion of fibration in this section and the standard ones in well known model category structures on the category of presheaves of groupoids: our notion is more geometric, in the sense that it does not distinguish between equivalent presheaves; in particular, any equivalence of presheaves of groupoids is a fibration in our sense.

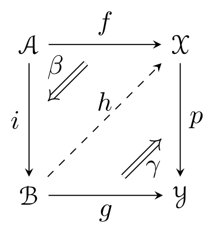
**Definition 7.1.** Let $i : \mathcal{A} \to \mathcal{B}$ and $p : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids. Then, $i$ has the *weak left lifting property (WLLP)* with respect to $p$ if given



there is a morphism $h : \mathcal{B} \to \mathcal{X}$, a 2-isomorphism $\gamma : g \Rightarrow p \circ h$ and a fiberwise homotopy $H$ from $f$ to $h \circ i$ relative to $\alpha \cdot (\gamma \circ i)$:



We say that $i$ has the *left lifting property (LLP)* with respect to $p$ if $H$ can be taken to be a ghost homotopy. In other words, there are 2-isomorphisms $\beta : f \Rightarrow h \circ i$ and $\gamma : g \Rightarrow p \circ h$ such that $p \circ \beta = \alpha \cdot (\gamma \circ i)$, i.e., the following diagram commutes ($\alpha$ is not shown in the diagram):



We say that $p$ has the *(weak) covering homotopy property* with respect to $\mathcal{A}$, if the inclusion $\mathcal{A} \to \mathcal{A} \times [0,1]$, $a \mapsto (a, 0)$, has (W)LLP with respect to $p$.

*Remark* 7.2. The usage of the term 'weak' (which means, 'up to fiberwise homotopy') in the above definition is in conflict with our usual usage of the term weak (which means, 'up to 2-isomorphism', as opposed to 'strict'). But since the above definition is quite standard in the homotopy theory literature, we deemed it inappropriate to change it. We apologize for the confusion this may cause.

**Definition 7.3** ([No14], Definitions 3.6, 3.7)**.** A morphism of presheaves of groupoids $p : \mathcal{X} \to \mathcal{Y}$ is called a *(weak) Serre fibration* if it has the (weak) covering homotopy property with respect to every finite CW complex $A$. That is, $A \to A \times [0,1]$ has the (W)LLP with respect to $p$. It is called a *(weak) trivial Serre fibration* if every finite CW inclusion $i : A \hookrightarrow B$ has the (W)LLP with respect to $p$.

**Lemma 7.4** (see [No14], Proposition 3.21)**.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a (weak) Serre fibration. Then, every cellular inclusion $i : A \hookrightarrow B$ of finite CW complexes that induces isomorphisms on all $\pi_n$ has the (W)LLP with respect to $p$.*

*Proof.* The map $i : A \hookrightarrow B$ being as above, $A$ becomes a deformation retract of $B$. Therefore, the map $i$ is a retract of the the map $j : B \to B \times [0,1]$, $j(b) = (b, 0)$,

$$\begin{array}{ccc} A & \xleftarrow{\;r\;} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xleftarrow[\;H\;]{} & B \times [0,1] \end{array}$$

Here, $r$ is the retraction and $H : B \times [0,1] \to B$ is a homotopy with $H_0 = r$ and $H_1 = \mathrm{id}_B$. Since $j$ has (W)LLP with respect to $p$, so does its retract $i$. □

*Remark* 7.5. As opposed to the notion of Reedy fibration that we will introduce in Definition 8.10, the notion of (weak) Serre fibration is "intrinsic" (or "geometric") in the sense that if $p : \mathcal{X} \to \mathcal{Y}$ is a (weak) Serre fibration and $p' : \mathcal{X}' \to \mathcal{Y}'$ is a morphism equivalent to it, then $p'$ is also a (weak) Serre fibration.

**Proposition 7.6.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks, and assume that $\mathcal{X}$ is Serre. Then, $p$ is a (weak) trivial Serre fibration if and only if it is a (weak) Serre fibration and a weak equivalence.*

*Proof.* By ([No14], Lemma 2.4), every morphism $p : \mathcal{X} \to \mathcal{Y}$ of topological stacks with $\mathcal{X}$ a Serre stack is a Serre morphism (in the sense of [No14], Definition 2.2). The result now follows from ([No14], Proposition 5.4). Note that ([No14], Proposition 5.4) is only stated for trivial Serre fibration, but it is also true for trivial weak Serre fibration; the first paragraph of the given proof (minus the last sentence) is in fact the proof of the statement for trivial weak Serre fibration. □

## 8. Reedy fibrations of stacks

In this section, we introduce Reedy fibrations between presheaves of groupoids (Definition 8.10) and establish some of their basic properties.

### 8.1. Model structure on Gpd.

Let Gpd denote the 2-category of groupoids. In this section we discuss the model structure on the underlying 1-category of Gpd.

**Definition 8.1.** Let $p : G \to H$ be a morphism in Gpd. We say that $p$ is a *fibration* if for any $x \in G$ and any isomorphism $\varphi : y \to p(x)$ in $H$, there exists an isomorphism $\psi : z \to x$ in $G$ such that $p(\psi) = \varphi$. In the literature, this is commonly referred to as an *isofibration.*

There is a model category structure on the category Gpd of groupoids where

- weak equivalences are equivalences of groupoids;
- cofibrations are maps that are injective on the set of objects;
- fibrations are as in Definition 8.1.

We refer the reader to ([Ho], Theorem 2.1) for more detail and further references.

**Lemma 8.2.** *Consider the following diagram in* Gpd*:*

$$\begin{array}{ccc} & & K \\ & & \downarrow{\scriptstyle p} \\ H & \longrightarrow & G \end{array}$$

*Suppose that $p$ is a fibration. Then, the natural map of groupoids*

$$H \times_G K \to H \tilde{\times}_G K$$

*is an equivalence.*

*Proof.* This functor is always fully faithful (see Section 3.5). It is straightforward that fibrancy of $p$ implies essential surjectivity. □

**Lemma 8.3.** *A morphism $i : G \to H$ in* Gpd *is a trivial cofibration if and only if it is essentially surjective and induces an isomorphism of groupoids between $G$ and a full subcategory of $H$. When this is the case, $G \times K \to H \times K$ is a trivial cofibration for every groupoid $K$.*

*Proof.* Straightforward. □

**Proposition 8.4.** *The above model structure on* Gpd *is left proper, simplicial, cofibrantly generated, combinatorial and monoidal (with respect to cartesian product).*

*Proof.* The properties left proper, simplicial and cofibrantly generated are proved in ([Ho], Theorem 2.1). Since Gpd is cofibrantly generated and locally presentable, it is, by definition, combinatorial.

To check that the model structure is monoidal we need to verify conditions $(i)$-$(iii)$ of ([Lu], Definition A.3.1.2). Conditions $(ii)$ and $(iii)$ are obvious. To check $(i)$ we have to show that the cartesian product $\times : \mathsf{Gpd} \times \mathsf{Gpd} \to \mathsf{Gpd}$ is a left Quillen bifunctor. That is, the following two conditions are satisfied:

(a) Let $i : A \to A'$ and $j : B \to B'$ be cofibrations in Gpd. Then, the induced map

$$i \wedge j : (A' \times B) \coprod_{A \times B} (A \times B') \to A' \times B'$$

is a cofibration in Gpd. Moreover, if either $i$ or $j$ is a trivial cofibration, then $i \wedge j$ is also a trivial cofibration.

(b) The cartesian product preserves small colimits separately in each variable.

The first part of (a) is easy as it only concerns the object sets of the groupoids in question, and the corresponding statement is true in the category of sets. To prove the second part of (a), assume that $i : A \to A'$ is a trivial cofibration. The claim follows from Lemma 8.3 and two-out-of-three applied to

$$A \times B' \to (A' \times B) \coprod_{A \times B} (A \times B') \to A' \times B'.$$

Condition (b) can be checked as follows (this argument was suggested to us by the referee). Let $K$ be an arbitrary groupoid. We have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Gpd}}(\operatorname*{colim}_{\alpha}(G_\alpha \times H), K) &\cong \lim_{\alpha} \mathrm{Hom}_{\mathsf{Gpd}}(G_\alpha \times H, K)\\ &\cong \lim_{\alpha} \mathrm{Hom}_{\mathsf{Gpd}}(G_\alpha, \mathrm{Hom}_{\mathsf{Gpd}}(H, K))\\ &\cong \mathrm{Hom}_{\mathsf{Gpd}}(\operatorname*{colim}_{\alpha} G_\alpha, \mathrm{Hom}_{\mathsf{Gpd}}(H, K))\\ &\cong \mathrm{Hom}_{\mathsf{Gpd}}((\operatorname*{colim}_{\alpha} G_\alpha) \times H, K).\end{aligned}$$

Thus, $\mathrm{colim}_\alpha(G_\alpha) \times H \cong (\mathrm{colim}_\alpha G_\alpha) \times H$. □

**Proposition 8.5.** *The model structure on* $\mathsf{Gpd}$ *is excellent in the sense of (*[Lu]*, Definition A.3.2.16).*

*Proof.* Axioms (A1)-(A4) of [ibid.] are straightforward to check. Axiom (A5), the Invertibility Hypothesis, follows from ([Lu], Lemma A.3.2.20) applied to the fundamental groupoid functor $\Pi_1 : \mathsf{sSet} \to \mathsf{Gpd}$. □

8.2. **Injective model structure on** $[\mathsf{C}^{op}, \mathsf{Gpd}]$**.** Let $\mathsf{C}$ be a small category. Since the model structure on $\mathsf{Gpd}$ is combinatorial (Proposition 8.4), by ([Lu], Proposition A.2.8.2) there is a model structure on the category $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$ of presheaves of groupoids, called the *injective model structure*, where

- weak equivalences are the objectwise weak equivalences as in Section 8.1;
- cofibrations are the objectwise cofibrations as in Section 8.1;
- fibrations have the right lifting property with respect to the trivial cofibrations.

We refer the reader to ([Lu], A.2.8) for more details on the injective model structure.

**Proposition 8.6.** *The injective model structure on* $[\mathsf{C}^{op}, \mathsf{Gpd}]$ *is* $\mathsf{Gpd}$*-enriched in the sense of (*[Lu]*, Definition A.3.1.5).*

*Proof.* This follows from ([Lu], Remark A.3.3.4). □

We are particularly interested in the cases $\mathsf{C} = \mathsf{Top}$ and $\mathsf{C} = \Delta$. In the case $\mathsf{C} = \Delta$, we have an explicit description of fibrations thanks to Proposition 8.9 below.

*Remark* 8.7. The smallness assumption on $\mathsf{C}$ is to allow us to quote results from Appendix A of [Lu]. As indicated at the beginning of Appendix A of [Lu], this is not a restrictive assumption as we can always fix a Grothendieck universe. For this reason, our treating $\mathsf{C} = \mathsf{Top}$ as a small category is not problematic.

8.3. **Reedy model structure on** $\mathsf{sGpd}$**.** The Reedy model structure on the category of simplicial groupoids $\mathsf{sGpd} = [\boldsymbol{\Delta}^{\mathrm{op}}, \mathsf{Gpd}]$ is defined as follows:

- weak equivalences are the objectwise weak equivalences;
- cofibrations are morphisms $X \to Y$ such that for every $n$ the map
$$L_nY \coprod_{L_nX} X_n \to Y_n$$
is a cofibration of groupoids (as in Section 8.1);
- fibrations are morphisms $X \to Y$ such that for every $n$ the map
$$X_n \to M_nX \times_{M_nY} Y_n$$
is a fibration of groupoids (as in Section 8.1).

Here, $L_nX$ stands for the latching object

$$L_nX := \operatorname*{colim}_{[k] \underset{\neq}{\twoheadrightarrow} [n]} X_k,$$

and $M_nX$ stands for the matching object

$$M_nX := \lim_{[n] \underset{\neq}{\hookrightarrow} [k]} X_k.$$

Let us unravel the above definitions. First of all, recall that the matching object $M_nX$ can be alternatively described by

$$M_nX = \mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, X),$$

where we regard the simplicial set $\partial\Delta^n$ as a simplicial groupoid. The map $X_n \to M_nX$ is the one induced by the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.

The Reedy fibration condition can now be restated as saying that

$$\mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, X) \to \mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, X) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, Y)} \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, Y)$$

is a fibration of groupoids.

**Lemma 8.8.** *Let $X$ and $Y$ be simplicial sets, regarded as objects in* $\mathsf{sGpd}$*. Then, any morphism $p : X \to Y$ is a Reedy fibration.*

*Proof.* This follows from the definition of a Reedy fibration and the fact that every map of sets, regarded as objects in $\mathsf{Gpd}$, is a fibration of groupoids. □

The Reedy cofibrations turn out to coincide with the objectwise cofibrations. That is, a morphism $p : X \to Y$ of simplicial groupoids is a Reedy cofibration if and only if $X_n \to Y_n$ is a cofibration of groupoids (in the sense of Section 8.1) for all $n$. This is a consequence of the following proposition.

**Proposition 8.9.** *The Reedy model structure and the injective model structure on* $\mathsf{sGpd} = [\mathbf{\Delta}^{op}, \mathsf{Gpd}]$ *coincide.*

*Proof.* We know that, by definition, the two model structures have the same weak equivalences. It remains to show that they have the same fibrations. Let $N : \mathsf{sGpd} = [\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Gpd}] \to [\mathbf{\Delta}^{\mathrm{op}}, \mathsf{sSet}]$ be the objectwise nerve functor, and let $\Pi_1$ be its left adjoint, the objectwise fundamental groupoid functor. We show that the following are equivalent:

(1) $p : X \to Y$ is a Reedy fibration in $\mathsf{sGpd}$.
(2) For all $n$, $N(X_n) \to M_n(N(X)) \times_{M_n(N(Y))} N(Y_n)$ is a Kan fibration of simplicial sets.
(3) $N(p) : N(X) \to N(Y)$ is a Reedy fibration in $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{sSet}]$.
(4) $N(p) : N(X) \to N(Y)$ is an injective fibration in $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{sSet}]$.
(5) $p : X \to Y$ is an injective fibration in $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Gpd}] = \mathsf{sGpd}$.

(1) $\Leftrightarrow$ (2) is true since the nerve functor preserves fiber products and $G \to H$ is a fibration of groupoids if and only if $N(G) \to N(H)$ is a Kan fibration. (2) $\Leftrightarrow$ (3) is true by definition. (3) $\Leftrightarrow$ (4) follows from the fact that injective model structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{sSet}]$ is the same as the Reedy model structure ([Lu], Example A.2.9.8 and Example A.2.9.21).

The implication (5) $\Rightarrow$ (4) follows from the fact that $\Pi_1$ takes a trivial cofibration of (presheaves of) simplicial sets to a trivial cofibration of (presheaves of) groupoids.

Finally, to prove (4) $\Rightarrow$ (5) we use the fact that $N$ preserves trivial cofibrations (for this use Lemma 8.3) and that $\Pi_1 \circ N = \mathrm{id}_{\mathsf{sGpd}}$. More precisely, to solve a lifting problem in $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Gpd}]$, we can first apply $N$, solve the lifting problem in $[\mathbf{\Delta}^{\mathrm{op}}, \mathsf{sSet}]$, and then apply $\Pi_1$ to obtain a solution to the original lifting problem. □

8.4. **Reedy fibrations in $\mathsf{pshGpd}$.** From now on, $\mathsf{C} = \mathsf{Top}$ (see Remark 8.7). We will use the notation $\mathsf{pshGpd}$ instead of $[\mathsf{C}^{\mathrm{op}}, \mathsf{Gpd}]$. We begin with our main definition.

**Definition 8.10.** We say that a map of presheaves of groupoids $p : \mathcal{X} \to \mathcal{Y}$ is a *Reedy fibration* if $p_{\mathbf{\Delta}} : \mathcal{X}_{\mathbf{\Delta}} \to \mathcal{Y}_{\mathbf{\Delta}}$ is a Reedy fibration in $\mathsf{sGpd}$ (see Section 8.3).

**Lemma 8.11.** *Let $X$ and $Y$ be presheaves of simplicial sets, regarded as objects in* $\mathsf{pshGpd}$*. Then, any morphism $p : X \to Y$ is a Reedy fibration.*

*Proof.* This follows from Lemma 8.8. $\square$

**Proposition 8.12.** *If $p : \mathcal{X} \to \mathcal{Y}$ is an injective fibration of presheaves of groupoids, then $p$ is a Reedy fibration.*

*Proof.* We have to show that, for every $n$, the map

$$\mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, \mathcal{X}_{\mathbf{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(\partial\Delta^n, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(\Delta^n, \mathcal{Y}_{\mathbf{\Delta}})$$

is a fibration of groupoids. Via the tilde construction, the above map is isomorphic to

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\Delta^n}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\partial\Delta^n}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\partial\Delta^n}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{\Delta^n}, \mathcal{Y}).$$

This map is a fibration of groupoids because $p : \mathcal{X} \to \mathcal{Y}$ is a fibration and $\widetilde{\partial\Delta^n} \to \widetilde{\Delta^n} = \Delta^n$ is a cofibration in the injective model structure on $\mathsf{pshGpd}$ (to see the latter, write $\partial\Delta^n$ as the colimit of its faces and use the fact that the tilde construction preserves colimits). The claim now follows from Proposition 8.6 (also see [Lu], Remark A.3.1.6(2')). $\square$

**Proposition 8.13.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a Reedy fibration of presheaves of groupoids, and let $A \to B$ be a monomorphism of simplicial sets. Then, the map*

$$\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

*and, equivalently (see Lemma 4.3), the map*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y})$$

*are fibrations of groupoids.*

*Proof.* In fact, the first map is a fibration of groupoids for any Reedy fibration $X \to Y$ in $\mathsf{sGpd}$ (in our case $X = \mathcal{X}_{\mathbf{\Delta}}$ and $Y = \mathcal{Y}_{\mathbf{\Delta}}$). In view of Proposition 8.9 this follows from Proposition 8.6 with $\mathsf{C} = \Delta$ (also see [Lu], Remark A.3.1.6(2')).

Alternatively, use ([Du], Lemma 4.5), with $M = \mathsf{Gpd}$, $K = A$, $L = B$, $X = \mathcal{X}_{\mathbf{\Delta}}$ and $Y = \mathcal{Y}_{\mathbf{\Delta}}$. $\square$

**Proposition 8.14.** *For any morphism of presheaves of groupoids $p : \mathcal{X} \to \mathcal{Y}$, there exists a strictly commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ p\ } & \mathcal{Y} \\ {\scriptstyle \sim}\downarrow{\scriptstyle g} & \nearrow_{p'} & \\ \mathcal{X}' & & \end{array}$$

*where $p'$ is an injective (hence, also Reedy) fibration and $g : \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ is an equivalence of presheaves of groupoids.*

*Proof.* Take the usual fibrant replacement in the injective model structure on $\mathsf{pshGpd}$ and use Proposition 8.12. $\square$

## 9. Singular functor for stacks

We shall define the functor $\mathrm{Sing} : \mathsf{pshGpd} \to \mathsf{sSet}$ of singular chains and establish some of its basic properties. This functor will be the focus of the rest of the paper.

9.1. **The functors $B$ and Sing.**

**Definition 9.1.** Let

$$B : \mathsf{pshGpd} \to \mathsf{bsSet},$$
$$\mathcal{X} \mapsto N(\mathcal{X}_{\boldsymbol{\Delta}}),$$

$$\mathrm{Sing} : \mathsf{pshGpd} \to \mathsf{sSet},$$
$$\mathcal{X} \mapsto \mathrm{Diag}(N(\mathcal{X}_{\boldsymbol{\Delta}})).$$

Here, $\mathsf{bsSet}$ stands for the category of bisimplicial sets, $N : \mathsf{sGpd} \to \mathsf{bsSet}$ is the levelwise nerve functor, and $\mathrm{Diag} : \mathsf{bsSet} \to \mathsf{sSet}$ refers to taking the diagonal of a bisimplicial set.

*Remark* 9.2. When restricted to $\mathsf{Top}$, the functor Sing coincides with the usual singular chains functor $\mathrm{Sing} : \mathsf{Top} \to \mathsf{sSet}$. More precisely, the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Top} & \longrightarrow & \mathsf{pshGpd} \\ & \searrow^{\mathrm{Sing}} & \downarrow \mathrm{Sing} \\ & & \mathsf{sSet} \end{array}$$

The top arrow in this diagram is the Yoneda embedding.

**Lemma 9.3.** *Let $f : X \to Y$ be a morphism of simplicial groupoids that induces equivalences of groupoids $X_n \to Y_n$ for all $n$. Then, the induced map $\mathrm{Diag}(NX) \to \mathrm{Diag}(NY)$ is a weak equivalence of simplicial sets.*

*Proof.* This follows from ([GoJa], Chapter IV, Proposition 1.7). □

**Corollary 9.4.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an equivalence of presheaves of groupoids. Then,* $\mathrm{Sing}(f) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ *is a weak equivalence of simplicial sets.*

We will need the following definition from ([GoJa], Chapter IV, Section 3.3).

**Definition 9.5.** Define the functor

$$d^* : \mathsf{sSet} \to \mathsf{bsSet}$$

to be the one uniquely determined by the following two properties:

- $d^*(\Delta^n) = \Delta^{n,n}$ (see Section 2 for notation);
- $d^*$ preserves colimits.

**Proposition 9.6.** *The functors* $\mathrm{Diag} : \mathsf{bsSet} \to \mathsf{sSet}$ *and* $N : \mathsf{sGpd} \to \mathsf{bsSet}$ *have left adjoints:*

$$\mathsf{sGpd} \underset{\Pi_1}{\overset{N}{\rightleftarrows}} \mathsf{bsSet} \underset{d^*}{\overset{\mathrm{Diag}}{\rightleftarrows}} \mathsf{sSet}.$$

*Here,* $\Pi_1$ *denotes the fundamental groupoid functor, and $d^*$ is as in (Definition 9.5). Therefore,* $\mathrm{Diag} \circ N$ *also has* $\Pi_1 \circ d^*$ *as left adjoint. In particular, the functors $N$,* Diag *and* $\mathrm{Sing} = \mathrm{Diag} \circ N \circ ()_{\boldsymbol{\Delta}}$ *preserve limits.*

*Proof.* For the first adjunction see ([Ho], Corollary 2.3). The second adjunction is discussed in ([GoJa], Chapter IV, Section 3.3). □

**Lemma 9.7.** *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be morphisms of presheaves of groupoids.*

(i) *If $\alpha : f \Rightarrow g$ is a 2-isomorphism, then we have an induced homotopy $\hat{\alpha}$ from* $\mathrm{Sing}(f)$ *to* $\mathrm{Sing}(g)$.

(ii) *If $h$ is a strict homotopy from $f$ to $g$ (see Definition 6.1), then we have an induced homotopy $\hat{h}$ from* $\mathrm{Sing}(f)$ *to* $\mathrm{Sing}(g)$.

*Proof.* Part $(ii)$ follows from the fact that Sing commutes with products (Proposition 9.6). To prove part $(i)$, let $\mathcal{I}$ be the constant presheaf of categories $\mathcal{I} : T \mapsto \{0 \to 1\}$, where $\{0 \to 1\}$ is the ordinal category (also denoted [1]). A 2-isomorphism $\alpha$ as above is the same thing as a morphism

$$\Phi_\alpha : \mathcal{X} \times \mathcal{I} \to \mathcal{Y}$$

whose restrictions to $\{0\}$ and $\{1\}$ are $f$ and $g$, respectively. It is easy to see that $\mathrm{Sing}(\mathcal{I}) = \Delta^1$. (Note that we have only defined Sing for presheaves of groupoids, but clearly the same definition makes sense for presheaves of categories as well.) By Proposition 9.6, we obtain a map of simplicial sets

$$\hat{\alpha} := \mathrm{Sing}(\Phi_\alpha) : \mathrm{Sing}(\mathcal{X}) \times \Delta^1 \to \mathrm{Sing}(\mathcal{Y}).$$

This is the desired homotopy. □

*Remark* 9.8.

(1) The operation $\alpha \mapsto \hat{\alpha}$ respects composition of 2-isomorphisms in the sense that $\widehat{\alpha \cdot \beta}$ is canonically homotopic to the "composition" of $\hat{\alpha}$ and $\hat{\beta}$. More precisely, $\hat{\alpha}$, $\hat{\beta}$ and $\widehat{\alpha \cdot \beta}$ are the three faces of a canonical map

$$\mathrm{Sing}(\mathcal{X}) \times \Delta^2 \to \mathrm{Sing}(\mathcal{Y}).$$

We also have higher coherences. That is, every string of $k$ composable 2-isomorphisms defines a canonical map

$$\mathrm{Sing}(\mathcal{X}) \times \Delta^k \to \mathrm{Sing}(\mathcal{Y})$$

whose restriction to various faces represent different ways of composing (a subset) of homotopies associated to these 2-isomorphisms.

(2) In the statement of Lemma 9.7$(ii)$ we could use a general homotopy $h = (H, \epsilon_0, \epsilon_1)$ from $f$ to $g$ (see Definition 6.1), but in this case instead of a homotopy from $f$ to $g$ we obtain a sequence of three composable homotopies $\hat{\epsilon_0}$, $\hat{\epsilon_1}$ and $\hat{h}$.

*Example* 9.9. In Lemma 9.10 below we will discuss the effect of Sing on 2-fiber products of presheaves of groupoids. To motivate the assumptions made there, we look at the following examples.

(1) The functor Sing does not respect 2-fiber products. For example, let $\mathcal{Z}$ be the constant presheaf on $\mathsf{Top}$ with value $J$ (viewed as a stack), where $J = \{0 \longleftrightarrow 1\}$ is the interval groupoid, and let $\mathcal{X} = \mathcal{Y} = *$ be singletons mapping to the points 0 and 1 in $\mathcal{Z}$, respectively. Then,

$$\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y} = * \tilde{\times}_{\mathcal{Z}} *$$

is equivalent to a point, while

$$\mathrm{Sing}(\mathcal{X}) \times_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) = * \times_{\mathrm{Sing}(\mathcal{Z})} *$$

is the empty set.

(2) It is not reasonable to expect that Sing takes 2-fiber products to homotopy fiber products either. For example, let $\mathcal{Z} = [0,1]$ be the unit interval, and let $\mathcal{X} = \mathcal{Y} = *$ be singletons mapping to the points 0 and 1 in $\mathcal{Z}$, respectively. Then,

$$\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y} = * \tilde{\times}_{\mathcal{Z}} * = * \times_{[0,1]} *$$

is the empty set, while

$$\mathrm{Sing}(\mathcal{X}) \overset{h}{\times}_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) = * \overset{h}{\times}_{\mathrm{Sing}(\mathcal{Z})} *$$

is non-empty (in fact, homotopy equivalent to a point).

**Lemma 9.10.** *Consider the following diagram in* pshGpd*:*

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow{\scriptstyle p} \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

*Suppose that $p$ is a Reedy fibration (by Lemma 8.11 this is automatic if $\mathcal{X}$ and $\mathcal{Z}$ are presheaves of sets). Then, there is a natural weak equivalence of simplicial sets*

$$\mathrm{Sing}(\mathcal{X}) \times_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{Sing}(\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y}).$$

*Proof.* Since $p$ is a Reedy fibration (hence objectwise fibration when restricted to $\mathbf{\Delta}$), the natural map

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y}$$

is an objectwise weak equivalence when restricted to $\mathbf{\Delta}$ (see Lemma 8.2). It follows from Lemma 9.3 that the induced map

$$\mathrm{Sing}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}) \xrightarrow{\sim} \mathrm{Sing}(\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y})$$

is a weak equivalence of simplicial sets. Precomposing with the isomorphism of Proposition 9.6, we obtain the desired weak equivalence

$$\mathrm{Sing}(\mathcal{X}) \times_{\mathrm{Sing}(\mathcal{Z})} \mathrm{Sing}(\mathcal{Y}) \xrightarrow{\cong} \mathrm{Sing}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}) \xrightarrow{\sim} \mathrm{Sing}(\mathcal{X} \tilde{\times}_{\mathcal{Z}} \mathcal{Y}).$$

□

## 10. Lifting lemmas

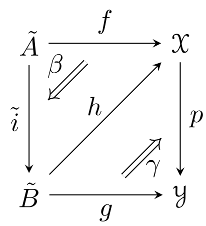
In this section we prove some lifting lemmas which will be used in the subsequent sections in the proofs of our main results. We invite the reader to consult Remark 7.2 before reading this section to prevent possible confusion caused by our usage of the term 'weak' in what follows.

10.1. **Strictifying lifts.** The following lemma is useful when we want to replace a lax solution to a strict lifting problem with a strict solution.

**Lemma 10.1.** *Consider the following strictly commutative diagram, where $p$ is a Reedy fibration of presheaves of groupoids (Definition 8.10) and $i$ is a monomorphism of simplicial sets:*

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{B} & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

*Suppose that there exists a lift $h$ and 2-isomorphisms $\beta$ and $\gamma$ making the following diagram 2-commutative:*



*Then, we can replace $h$ by a 2-isomorphic morphism $h'$ so that $\beta$ and $\gamma$ become the identity 2-isomorphisms. More precisely, $h' \circ \tilde{i} = f$, $p \circ h' = g$, and there is $\theta : h' \Rightarrow h$ such that $\theta \circ \tilde{i} = \beta$ and $p \circ \theta = \gamma$.*

*Proof.* By Proposition 8.13, the natural map

$$\Psi : \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y})$$

is a fibration of groupoids. The map $h$ can be regarded as an object on the left hand side, with $\Psi(h) = (h \circ \tilde{i}, p \circ h)$. Since $\Psi$ is a fibration, we can lift the 2-isomorphism $(\beta, \gamma) : (f, g) \Rightarrow (h \circ \tilde{i}, p \circ h)$ to a 2-isomorphism $\theta : h' \to h$. This is exactly what we need. □

In most of our applications of the above lemma, we will have $B = \Delta^n$, in which case $\tilde{B} = |\Delta^n|$.

**Corollary 10.2.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a Reedy fibration of presheaves of groupoids, $A$ a simplicial set, and $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ a restricted homotopy (see Section 6.2) relative to $\mathcal{Y}$ starting at $H_0 := H|_{\tilde{A} \times \{0\}} : \tilde{A} \to \mathcal{X}$. Then, for every 2-isomorphism $\beta : f' \Rightarrow H_0$, there exists a restricted homotopy*

$$H' : \widetilde{A \times \Delta^1} \to \mathcal{X} \text{ relative to } \mathcal{Y}$$

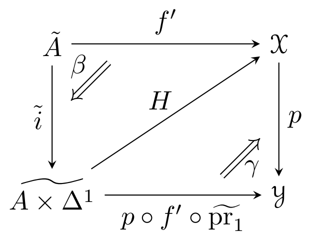
*and a 2-isomorphism $\Theta : H' \Rightarrow H$ such that $p \circ \Theta = p \circ \beta \circ \widetilde{\mathrm{pr}_1}$ as 2-isomorphisms*

$$p \circ f' \circ \widetilde{\mathrm{pr}_1} \Rightarrow p \circ H_0 \circ \widetilde{\mathrm{pr}_1} \, (= p \circ H)$$

*(i.e., $\Theta$ is relative to $p \circ \beta \circ \widetilde{\mathrm{pr}_1}$), and that*

$$f' = H'_0 := H'|_{\tilde{A} \times \{0\}} \text{ and } \beta = \Theta_0 := \Theta|_{\tilde{A} \times \{0\}}.$$

*Proof.* With the notation of Lemma 10.1, let $B = A \times \Delta^1$, $i : A \to A \times \Delta^1$ the inclusion at time 0, $f = f'$, $g = p \circ f' \circ \widetilde{\mathrm{pr}_1}$, $h = H$, $\beta = \beta$ and $\gamma = p \circ \beta \circ \widetilde{\mathrm{pr}_1}$, as in the diagram




The result now follows from Lemma 10.1. □

10.2. **Strict lifts for Serre+Reedy fibrations.** From now on, we will assume that our simplicial sets $A$ and $B$ are finite non-singular simplicial sets (see Corollary 5.6 and the preceding paragraph). For example, $\Delta^n$, $\partial\Delta^n$ and $\Lambda^n_k$ have this property. If $A$ and $B$ have this property, then $A \times B$ also has this property, and so does any colimit $A \coprod_C B$, as long as the maps $C \to A$ and $C \to B$ are monomorphisms. In particular, if $i : A \to B$ is a monomorphism, then the mapping cylinder $\mathrm{Cyl}(i)$ has this property.

**Lemma 10.3.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks and $i : A \to B$ a monomorphism of finite non-singular simplicial sets. If $p$ is a (weak) Serre fibration and either $p$ or $i$ is a weak equivalence, then $\tilde{i} : \tilde{A} \to \tilde{B}$ has (weak) LLP with respect to $p$ (see Definition 7.1).*

*Proof.* Consider the lifting problem

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \tilde{i}}\downarrow & \swArrow \alpha & \downarrow{\scriptstyle p} \\ \tilde{B} & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

First note that to solve it we are allowed to replace each of $f$ and $g$ with a 2-isomorphic morphism (and adjust $\alpha$ accordingly). So, we may assume, by Corollary 5.6, that there are maps $f' : |A| \to \mathcal{X}$ and $g' : |B| \to \mathcal{Y}$ such that $f = f' \circ \psi_A$ and $g = g' \circ \psi_B$. Here, $\psi_A : \tilde{A} \to |A|$ is as in Eq. (4.1).Thus, our lifting problem translates to

$$\begin{array}{ccc} |A| & \xrightarrow{f'} & \mathcal{X} \\ {\scriptstyle |i|}\downarrow & \swArrow \alpha' & \downarrow{\scriptstyle p} \\ |B| & \xrightarrow[g']{} & \mathcal{Y} \end{array}$$

(The existence of the unique $\alpha'$ is guaranteed by Corollary 5.6.) This problem can now be solved under the given assumptions. Precomposing with the $\psi$ maps, we obtain a solution to the original lifting problem. (Also see Proposition 7.6.) □

**Lemma 10.4.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks and $i : A \to B$ a monomorphism of finite non-singular simplicial sets. If $p$ is a Serre fibration and also a Reedy fibration, and either $p$ or $i$ is a weak equivalence, then $\tilde{i} : \tilde{A} \to \tilde{B}$ has strict LLP with respect to $p$. That is, if in the diagram*

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \tilde{i}}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \tilde{B} & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

*the outer square is strictly commutative, then there exists a lift $h$ making both triangles strictly commutative.*

*Proof.* First use Lemma 10.3 to find a solution $h$ which makes the two triangles commutative up to 2-isomorphism. Then use Lemma 10.1 to rectify $h$ to make the triangles strictly commutative. □

**Corollary 10.5.** *Assumptions being as in Lemma 10.4, the map*

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y})$$

*and, equivalently (see Lemma 4.3), the map*

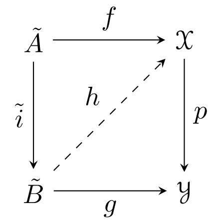
$$\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\boldsymbol{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\boldsymbol{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\boldsymbol{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\boldsymbol{\Delta}})$$

*are fibrations of groupoids that are surjective on objects (hence, also on morphisms).*

*Proof.* Surjectivity on objects is simply a restatement of Lemma 10.4. They are fibrations by Proposition 8.13. □

10.3. **Strict lifts for weak Serre+Reedy fibrations.**

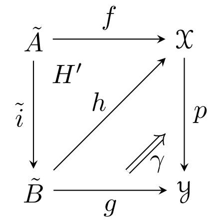
**Lemma 10.6.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks and $i : A \to B$ a monomorphism of finite non-singular simplicial sets. If $p$ is a weak Serre fibration and also a Reedy fibration, and either $p$ or $i$ is a weak equivalence, then $\tilde{i} : \tilde{A} \to \tilde{B}$ has* strict *WLLP with respect to $p$ in the following sense. If in the diagram*



*the outer square is strictly commutative, then there exists a lift $h$ and a morphism $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ such that*

i) *the lower triangle is strictly commutative, and*
ii) *$H$ is a strict restricted fiberwise homotopy from $f$ to $h \circ \tilde{i}$ relative to $\mathcal{Y}$ (see Section 6.2), where strictness means that $H_0 = f$ and $H_1 = h \circ \tilde{i}$.*

*Proof.* First use Lemma 10.3 to find a solution $h$ which makes the lower triangle commutative up to a 2-isomorphism and the upper triangle commutative up to fiberwise homotopy $H' : \tilde{A} \times [0,1] \to \mathcal{X}$, as in the diagram

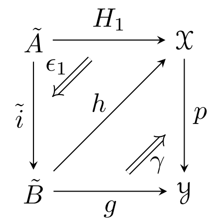


First, we rectify $H'$ using Corollary 10.2, as follows. (Note that Corollary 10.2 only applies to restricted homotopy and not ordinary homotopy, so we need to replace $H'$ by the corresponding restricted homotopy $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$; see Lemma 6.6.) Since $p$ is a Reedy fibration and $\widetilde{A \times \Delta^1}$ is cofibrant, Proposition 8.13 (with $B = A \times \Delta^1$ and $A$ the empty set) implies that

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X}) \to \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{Y})$$

is a fibration of groupoids. So, we can replace $H$ by a 2-isomorphic map so that it becomes relative to $\mathcal{Y}$ (namely, $p \circ f \circ \widetilde{\mathrm{pr}}_1 = p \circ H$ on the nose); see Remark 6.5 to see why this is allowed. We can now use Corollary 10.2 to rectify $H$ so that $H_0 = f$.

There are two more things to do now: ensure that the 2-isomorphism $\epsilon_1 : H_1 \Rightarrow h \circ \tilde{i}$ becomes an equality, and that $\gamma = \mathrm{id}$. This is achieved by applying Lemma 10.1 to the diagram



to adjust $h$ so that $\epsilon_1$ and $\gamma$ become the identity 2-isomorphisms. □

Let $p : \mathcal{X} \to \mathcal{Y}$ be a map of presheaves of groupoids and $i : A \to B$ a map of simplicial sets. Let $L$ be the groupoid

$$\begin{aligned} L :=\ & \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X})_p \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \\ \cong\ & \mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{X}_{\boldsymbol{\Delta}})_p \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A,\mathcal{X}_{\boldsymbol{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\boldsymbol{\Delta}}) \end{aligned}$$

of pairs $(H, h)$, where $h : \tilde{B} \to \mathcal{X}$ is a morphism and $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ is a restricted fiberwise homotopy relative to $\mathcal{Y}$ such that $H_1 = h \circ \tilde{i}$. Here $H_1 : \tilde{A} \to \mathcal{X}$ stands for the precomposition of $H$ with the time 1 inclusion map $\tilde{A} \to \widetilde{A \times \Delta^1}$, and the subscript $p$ stands for 'fiberwise relative to $\mathcal{Y}$'. More precisely,

$$\begin{aligned} \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X})_p :=\ & \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A\times\Delta^1},\mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y}) \\ \cong\ & \mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{X}_{\boldsymbol{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A\times\Delta^1,\mathcal{Y}_{\boldsymbol{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\boldsymbol{\Delta}}), \end{aligned}$$

where the first map in the fiber product is induced by $p$, and the second map is induced by the projection $\widetilde{A \times \Delta^1} \to \tilde{A}$.

Thus, we have isomorphisms of groupoids

$$\begin{aligned} L \cong\ & \mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A \times \Delta^1}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\widetilde{A\times\Delta^1},\mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{Y}) \\ & \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{X})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{X}) \\ \cong\ & \mathrm{Hom}_{\mathsf{sGpd}}(A \times \Delta^1, \mathcal{X}_{\boldsymbol{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A\times\Delta^1,\mathcal{Y}_{\boldsymbol{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\boldsymbol{\Delta}}) \\ & \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A,\mathcal{X}_{\boldsymbol{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\boldsymbol{\Delta}}). \end{aligned}$$

**Corollary 10.7.** *Notation being as above and assumptions being as in Lemma 10.6, the map*

$$\begin{aligned} L &\to \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y}) \\ (H, h) &\mapsto (H_0, p \circ h) \end{aligned}$$

*and, equivalently (see Lemma 4.3), the map*

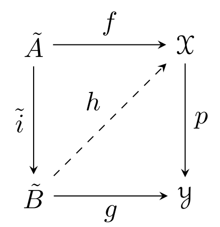
$$L \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\boldsymbol{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A,\mathcal{Y}_{\boldsymbol{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\boldsymbol{\Delta}})$$

*are fibrations of groupoids that are surjective on objects (hence, also on morphisms, as well as tuples of composable morphisms).*

*Proof.* Let us denote the map in question by $\Psi$. The surjectivity of $\Psi$ on objects is simply a restatement of Lemma 10.6. Let us spell this out. Consider an object in

$$\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{A},\mathcal{Y})} \mathrm{Hom}_{\mathsf{pshGpd}}(\tilde{B}, \mathcal{Y}),$$

namely a pair $(f, g)$ making the outer square in the following diagram strictly commutative:




By Lemma 10.6, this lifting problem has a weak solution $(H, h)$, namely $h : \tilde{B} \to \mathcal{X}$ and $H : \widetilde{A \times \Delta^1} \to \mathcal{X}$ such that

i) the lower triangle is strictly commutative, and
ii) $H$ is a strict restricted fiberwise homotopy from $f$ to $h \circ \tilde{i}$ relative to $\mathcal{Y}$, where strictness means that $H_0 = f$ and $H_1 = h \circ \tilde{i}$.

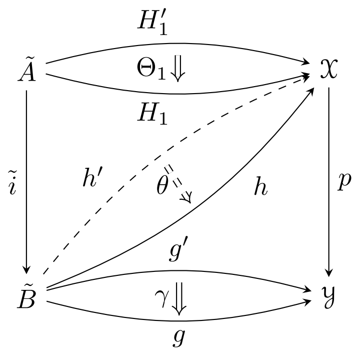
By definition of $L$, such a pair determines an object in $L$ mapping to the pair $(f,g)$, $\Psi(H,h)=(f,g)$. This proves surjectivity on objects.

To prove fibrancy, suppose in the above setting that we are also given 2-isomorphisms $\beta : f' \Rightarrow f$ and $\gamma : g' \Rightarrow g$ such that $p \circ \beta = \gamma \circ \tilde{i}$. We need to construct a pair $(\Theta,\theta) \in \mathrm{Mor}(L)$ with the following properties:

i) $\theta : h' \Rightarrow h$ is relative to $\gamma$ (that is, $p \circ \theta = \gamma$),
ii) $\Theta : H' \Rightarrow H$ is relative to $\gamma \circ \tilde{i} \circ \widetilde{\mathrm{pr}_1} = p \circ \beta \circ \widetilde{\mathrm{pr}_1}$ (that is, $p \circ \Theta = \gamma \circ \tilde{i} \circ \widetilde{\mathrm{pr}_1}$), $\Theta_0 = \beta$ and $\Theta_1 = \theta \circ \tilde{i}$.

By Corollary 10.2, we have a restricted fiberwise homotopy $H' : \widetilde{A \times \Delta^1} \to \mathcal{X}$ relative to $\mathcal{Y}$, and a 2-isomorphism $\Theta : H' \Rightarrow H$ relative to $p \circ \beta \circ \widetilde{\mathrm{pr}_1}$ such that $f' = H'_0$ and $\beta = \Theta_0$. This is our desired $\Theta$.

To find $\theta$, note that its restriction to $\tilde{A}$ is already determined, namely $\Theta_1$. So, we need to extend $\Theta_1$ to the whole of $\tilde{B}$ in such a way that $p \circ \theta = \gamma$. We do this by solving the following lifting problem for $(h',\theta)$:




Existence of a solution is guaranteed by Proposition 8.13. □

## 11. Singular functor preserves fibrations

In this section we study the effect of the functor Sing on fibrations of stacks. We begin with a simple example to show why the Reedy condition is necessary in the statement of our main result (Theorem 11.8).

*Example* 11.1. Let $X$ be a trivial groupoid with more than one point, namely one that is equivalent but not equal to a point. Let $\mathcal{X}$ be the constant presheaf with value $X$ (viewed as a stack). Pick a point in $X$ and consider the map $* \to \mathcal{X}$. This map is an equivalence of stacks, hence is a Serre fibration. However, the induced map of simplicial sets

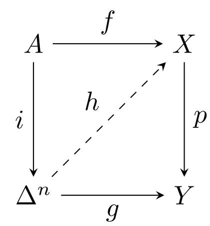
$$\mathrm{Sing}(*) = * \to N(X) = \mathrm{Sing}(\mathcal{X})$$

is not a Kan fibration.

### 11.1. Weak Kan fibrations.

In what follows, the homotopy groups $\pi_n(X,x)$ of a simplicial set $X$ which is not necessarily Kan are taken to be those of its geometric realization.

**Definition 11.2.** We say that a map of simplicial sets $p : X \to Y$ is a *weak Kan fibration* if for any trivial cofibration $i : A \to \Delta^n$, every lifting problem



has a weak solution; namely, there exists $h : \Delta^n \to X$ such that the bottom triangle commutes and $f : A \to X$ is fiberwise homotopic to $h \circ i : A \to X$ relative to $Y$. We say that $p$ is a *weak trivial Kan fibration* if it is a weak Kan fibration and, in addition, it has the weak lifting property with respect to the inclusions $\partial\Delta^n \to \Delta^n$, $n \geq 0$.

In the above definition, a fiberwise homotopy relative to $Y$ means a map of simplicial sets $H : A \times \Delta^1 \to X$ such that $p \circ H$ is the trivial homotopy from $p \circ f$ to itself.

*Remark* 11.3. We do not know if the above definition is the "correct" simplicial counterpart of the notion of a weak Serre fibration, but it serves our purposes in this paper (thanks to Lemma 11.6). It is not clear to us whether a weak Kan fibration will have the weak left lifting property with respect to *all* trivial cofibrations.

**Lemma 11.4.** *Let $p : X \to Y$ be a trivial weak Kan fibration. Assume that $Y$ is a Kan simplicial set, and that there exists a Kan simplicial set $X'$ together with a weak equivalence $X' \to X$. Then, $p$ is a weak equivalence.*

*Proof.* First we prove that $\pi_n(p)$ is injective. Let $x$ be a base point that is in the image of $X'$, and let $y = p(x)$. The fact that $X'$ is Kan guarantees that any class in $\pi_n(X, x)$ is represented by a pointed map $f : \partial\Delta^n \to X$. If the image of this class in $\pi_n(Y, y)$ is trivial, we will have, since $Y$ is Kan, a filling $g$ for $p \circ f$, as in the diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\ f\ } & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[\ g\ ]{} & Y \end{array}$$

So, a lift $h$ exists which makes the diagram commutative (possibly after replacing $f$ by a fiberwise homotopic map). This implies that the class represented by $f$ in $\pi_n(X, x)$ is trivial.

To prove surjectivity of $\pi_n(p)$, let $g : \partial\Delta^n \to Y$ represent an arbitrary class in $\pi_n(Y, y)$. To lift this to $X$, we begin by lifting $g|_{\Lambda_0^n} : \Lambda_0^n \to Y$ to $X$. To do so, first extend $g|_{\Lambda_0^n}$ to the whole $\Delta^n$ using the Kan property of $Y$. Then, apply the weak lifting property to the trivial cofibration $\{0\} \to \Delta^n$. Restricting the outcome to $\Lambda_0^n$, we find a lift $\hat{g} : \Lambda_0^n \to X$, sending 0 to a point that is fiberwise homotopic to $x$. Since $\pi_n(X, x)$ and $\pi_n(X, \hat{g}(0))$ have the same image in $\pi_n(Y, y)$, as can be seen by passing to the geometric realization, there is no harm in replacing $x$ with $\hat{g}(0)$. So we may assume that $\hat{g}(0) = x$. Now consider the following lifting problem

$$\begin{array}{ccc} \partial\Delta^{n-1} & \xrightarrow{\ \hat{g} \circ (d_0|_{\partial\Delta^{n-1}})\ } & X \\ {\scriptstyle j}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \Delta^{n-1} & \xrightarrow[\ g \circ d_0\ ]{} & Y \end{array}$$

Here, $d_0 : \Delta^{n-1} \to \partial\Delta^n$ is the $0^{th}$ face of $\partial\Delta^n$ and $j : \partial\Delta^{n-1} \to \Delta^{n-1}$ is the inclusion map. A weak solution to this problem can be glued to $\hat{g}$ to give a map

$$G : \Lambda_0^n \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1} \times \Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1} \longrightarrow X$$

making the following diagram commutative

$$\begin{array}{ccc} \Lambda^n_0 \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1}\times\Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1} & \xrightarrow{G} & X \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle p} \\ \partial\Delta^n & \xrightarrow[g]{} & Y \end{array}$$

Here,

$$P : \Lambda^n_0 \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1}\times\Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1} \to \partial\Delta^n$$

is the map that collapses $\partial\Delta^{n-1}\times\Delta^1$ to $\partial\Delta^{n-1}$ via the first projection; note that

$$|\Lambda^n_0 \coprod_{\partial\Delta^{n-1}} (\partial\Delta^{n-1}\times\Delta^1) \coprod_{\partial\Delta^{n-1}} \Delta^{n-1}|$$

is homeomorphic to an $n$-sphere. The (geometric realization of the) map $G$ represents a lift of the class in $\pi_n(Y,y)$ represented by $g$ to a class in class in $\pi_n(X,x)$. This completes the proof of surjectivity. □

**Lemma 11.5.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks that is a (weak) (trivial) Serre fibration and a Reedy fibration. Let $R_0(\mathcal{X}) = \mathrm{Ob}(\mathcal{X}_{\mathbf{\Delta}})$, $R_1(\mathcal{X}) = \mathrm{Mor}(\mathcal{X}_{\mathbf{\Delta}})$ and*

$$R_m(\mathcal{X}) = R_1(\mathcal{X}) \times_{R_0(\mathcal{X})} \times \cdots \times_{R_0(\mathcal{X})} R_1(\mathcal{X}).$$

*Then, for every $m \geq 0$, the induced map*

$$R_m(\mathcal{X}) \to R_m(\mathcal{Y})$$

*is a (weak) (trivial) Kan fibration of simplicial sets.*

*Proof.* First, we prove the statement in the case of a Serre fibration. Let $A = \Lambda^n_k$ and $B = \Delta^n$, and let $i : A \to B$ be the horn inclusion. By Corollary 10.5, we have a fibration of groupoids

$$\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \to \mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{\mathrm{Hom}_{\mathsf{sGpd}}(A,\mathcal{Y}_{\mathbf{\Delta}})} \mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

which is surjective on objects. Taking nerves on both sides, we find a fibration of simplicial sets

$$N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}) \to N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A,\mathcal{Y}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}})$$

which is surjective on $m$-simplices, for all $m$. The surjectivity on $m$-simplices precisely translates to the fact that $i$ has LLP with respect to $R_m(\mathcal{X}) \to R_m(\mathcal{Y})$, as the above map on the level on $m$-simplices is, term by term, equal to the map

$$\mathrm{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})) \to \mathrm{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X})) \times_{\mathrm{Hom}_{\mathsf{sSet}}(A,R_m(\mathcal{Y}))} \mathrm{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{Y})).$$

This shows that $R_m(\mathcal{X}) \to R_m(\mathcal{Y})$ is a Kan fibration. The case of a trivial Serre fibration is proved similarly (taking $A = \partial\Delta^n$ instead of $\Lambda^n_k$).

Now consider the case where $p$ is a weak Serre fibration. Let $B = \Delta^n$ and $i : A \to B$ be as in Definition 11.2. By Corollary 10.7, we have a fibration of simplicial sets

$$NL \to N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{X}_{\mathbf{\Delta}}) \times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A,\mathcal{Y}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{Y}_{\mathbf{\Delta}}) \tag{11.1}$$

which is surjective on $m$-simplices, for all $m$. By the discussion just before Lemma 10.7, and the fact that taking nerves commutes with fiber products, $NL$ is isomorphic to

$$N\,\mathrm{Hom}_{\mathsf{sGpd}}(A\times\Delta^1, \mathcal{X}_{\mathbf{\Delta}}) \times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A\times\Delta^1,\mathcal{Y}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(A, \mathcal{Y}_{\mathbf{\Delta}})$$
$$\times_{N\,\mathrm{Hom}_{\mathsf{sGpd}}(A,\mathcal{X}_{\mathbf{\Delta}})} N\,\mathrm{Hom}_{\mathsf{sGpd}}(B, \mathcal{X}_{\mathbf{\Delta}}).$$

Its set of $m$-simplices is then equal to

$$\begin{aligned}(NL)_m =&\operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X})) \times_{\operatorname{Hom}_{\mathsf{Set}}(A\times\Delta^1, R_m(\mathcal{Y}))} \operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{Y})) \\ &\qquad\qquad\qquad\qquad \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})) \\ \cong& \operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X}))_p \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})),\end{aligned}$$

where

$$\operatorname{Hom}_{\mathsf{sSet}}(A\times\Delta^1, R_m(\mathcal{X}))_p := \operatorname{Hom}_{\mathsf{sSet}}(A\times\Delta^1, R_m(\mathcal{X}))\times_{\operatorname{Hom}_{\mathsf{Set}}(A\times\Delta^1, R_m(\mathcal{Y}))}\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{Y}))$$

is the set of fiberwise homotopies. Thus, we can think of $(NL)_m$ as the set of pairs $(H, h)$, where $h : B \to R_m(\mathcal{X})$ is a map of simplicial sets and $H : A \times \Delta^1 \to R_m(\mathcal{X})$ is a fiberwise homotopy relative to $R_m(\mathcal{Y})$ such that $H_1 = h \circ i$.

Hence, on the level of $m$-simplices, the map 11.1 above can be identified with the natural map

$$\begin{aligned}&\operatorname{Hom}_{\mathsf{sSet}}(A \times \Delta^1, R_m(\mathcal{X}))_p \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{X})) \\ &\to \operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{X})) \times_{\operatorname{Hom}_{\mathsf{sSet}}(A, R_m(\mathcal{Y}))} \operatorname{Hom}_{\mathsf{sSet}}(B, R_m(\mathcal{Y})).\end{aligned}$$

which assigns to any weak solution $(H, h)$, viewed as an element in the left hand side, its associated lifting problem $(f, g)$, viewed as an element in the right hand side, as in the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & R_m(\mathcal{X}) \\ {\scriptstyle i}\downarrow & H \quad \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & R_m(\mathcal{Y}) \end{array}$$

The surjectivity of this map precisely means that any such lifting problem has a weak solution.

The case of a weak trivial Serre fibration is proved similarly. □

11.2. **A lemma on $d^* : \mathsf{sSet} \to \mathsf{bsSet}$.** In this section we prove a lemma which is used in the proof of Lemma 11.7, which in turn plays an important role in the proof of our first main result, Theorem 11.8.

First, we briefly recall notion of exterior product of simplicial sets. Given simplicial sets $X$ and $Y$, their *exterior product* is the bisimplicial set $X \boxtimes Y$ defined by

$$(X \boxtimes Y)_{m,n} := X_m \times Y_n.$$

We have $\operatorname{Diag}(X \boxtimes Y) = X \times Y$. The exterior product has the property that the functor

$$\begin{aligned}\mathsf{sSet} &\to \mathsf{bsSet}, \\ A &\mapsto A \boxtimes \Delta^n,\end{aligned}$$

is left adjoint to

$$\begin{aligned}\mathsf{bsSet} &\to \mathsf{sSet}, \\ X &\mapsto X_{*,n}.\end{aligned}$$

Let $A \to B$ be a map of simplicial sets. Recall the functor $d^* : \mathsf{sSet} \to \mathsf{bsSet}$ from Definition 9.5. We have a natural map

$$d^*(A) \to A \boxtimes B,$$

namely, the adjoint (see Proposition 9.6) to the diagonal inclusion

$$A \to \operatorname{Diag}(A \boxtimes B) = A \times B.$$

In the next lemma we show that for any monomorphism $A \to \Delta^n$, the map $d^*(A) \to A \boxtimes \Delta^n$ is a trivial cofibration. The case $A = \Lambda^n_k$ of the following lemma is proved in [GoJa] (see [GoJa], top of the page 221, just before Lemma 3.12).

**Lemma 11.6.** *Let $\gamma : A \to \Delta^n$ be a cofibration (not necessarily trivial) of simplicial sets. Then, for every $m$, we have*

$$(d^*A)_{m,*} = \coprod_{\alpha \in A_m} C_\alpha,$$

*where $C_\alpha \subseteq A$ is the union of all faces of $A$ that contain $\alpha$. The natural map of bisimplicial sets*

$$i : d^*(A) \to A \boxtimes \Delta^n,$$

*namely, the left adjoint to the diagonal inclusion*

$$(\mathrm{id}, \gamma) : A \to \mathrm{Diag}(A \boxtimes \Delta^n) = A \times \Delta^n,$$

*is given on the $m^{th}$ column by the inclusion*

$$i_m : \coprod_{\alpha \in A_m} C_\alpha \hookrightarrow \coprod_{\alpha \in A_m} \Delta^n.$$

*In particular, $i_m$ is a trivial cofibration of simplicial sets for every $m$ (thus, $i$ is a vertical pointwise trivial cofibration of bisimplicial sets).*

In the above lemma, by a *face* of $A$ we mean the sub simplicial set generated by a (non-degenerate) simplex in $A$. Note that such a face is isomorphic to some simplex $\Delta^m$ and that $A \subseteq \Delta^n$ is necessarily a union of a collection of faces in $\Delta^n$.

*Proof.* First we consider the case where $\gamma$ is the inclusion of a face (so $A = \Delta^d$ for some $d$). In this case, $d^*(A) = A \boxtimes A$, which can be identified with a sub bisimplicial set of $A \boxtimes \Delta^n$ via the map whose effect on the $m^{th}$ column is given by

$$i^A_m : \coprod_{\alpha \in A_m} A \overset{\gamma}{\hookrightarrow} \coprod_{\alpha \in A_m} \Delta^n \ (\subseteq \coprod_{\alpha \in \Delta^n_m} \Delta^n).$$

The key observation here is that, for any two faces $F_i$ and $F_j$ of $A$, the image of $i^{F_i \cap F_j}_m$ in the $m^{\text{th}}$ column $\coprod_{\alpha \in \Delta^n_m} \Delta^n$ of $\Delta^n \boxtimes \Delta^n$ is equal to the intersection of the images of $i^{F_j}_m$ and $i^{F_k}_m$.

Now, for general $A$, write is as a coequalizer of the inclusions of its faces, namely

$$A = \mathrm{coeq}\left( \coprod_{j,k} F_j \cap F_k \rightrightarrows \coprod_j F_j \right).$$

Since $d^*$ commutes with colimits, we have

$$d^*(A) = \mathrm{coeq}\left( \coprod_{j,k} d^*(F_j \cap F_k) \rightrightarrows \coprod_j d^*(F_j) \right).$$

The observation above that $i_m$ respects intersections implies that $i^A_m : d^*(A) \to \Delta^n \boxtimes \Delta^n$ is injective and the image under $i^A_m$ of $d^*(A)$ in the $m^{\text{th}}$ column $\coprod_{\alpha \in \Delta^n_m} \Delta^n$ of $\Delta^n \boxtimes \Delta^n$ is the union of images of all $i^{F_j}_m$. This is precisely $\coprod_{\alpha \in A_m} C_\alpha$. ☐

11.3. **A criterion for diagonal fibrations.** To prove our first main result we need a generalization of Lemma 4.8 of ([GoJa], Chapter IV) which we now prove.

In the next lemma, we are regarding $X$ as the simplicial object $[m] \mapsto X_{m,*}$ in $\mathsf{sSet}$.

**Lemma 11.7.** *Let $f : X \to Y$ be a Reedy fibration of bisimplicial sets. Let $\gamma : A \to \Delta^n$ be a monomorphism. Suppose that $\gamma$ has (weak) left lifting property with respect to $f_{*,n} : X_{*,n} \to Y_{*,n}$, for all $n$ (Definition 11.2). Then, $\gamma$ has (weak) left lifting property with respect to* $\mathrm{Diag}(f) : \mathrm{Diag}(X) \to \mathrm{Diag}(Y)$. *In particular, if each $f_{*,n} : X_{*,n} \to Y_{*,n}$ is a (weak) (trivial) Kan fibration, then so is* $\mathrm{Diag}(f)$.

*Proof.* We want to show that $\gamma : A \to \Delta^n$ has (W)LLP with respect to $\mathrm{Diag}(f)$. By adjunction, this is equivalent to showing that the lifting problem

$$(*) \qquad \begin{array}{ccc} d^*(A) & \xrightarrow{\ u\ } & X \\ {\scriptstyle d^*(\gamma)}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle f} \\ d^*(\Delta^n) & \xrightarrow[\ v\ ]{} & Y \end{array}$$

in bisimplicial sets has a solution, with the caveat that, in the 'weak' setting, instead of a fiberwise homotopy in the upper triangle of $(*)$ we should be asking for a map $d^*(A \times \Delta^1) \to X$ (with the obvious properties).

We solve $(*)$ in two steps, by writing the left vertical map

$$d^*(A) \to d^*(\Delta^n) = \Delta^{n,n} = \Delta^n \boxtimes \Delta^n$$

as composition of two inclusions

$$d^*(A) \xrightarrow{i} A \boxtimes \Delta^n \xrightarrow{j} \Delta^n \boxtimes \Delta^n.$$

Here, the map $i$ is adjoint to the diagonal inclusion

$$(\mathrm{id}, \gamma) : A \to \mathrm{Diag}(A \boxtimes \Delta^n) = A \times \Delta^n;$$

see the paragraph before Lemma 11.6.

**Step 1.** We first solve the lifting problem

$$(**) \qquad \begin{array}{ccc} d^*(A) & \xrightarrow{\ u\ } & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle f} \\ A \boxtimes \Delta^n & \xrightarrow[\ v \circ j\ ]{} & Y \end{array}$$

By Lemma 11.6, $i$ is a pointwise trivial cofibration, so it has strict LLP with respect to $f$, as $f$ is a Reedy fibration (see [GoJa], Chapter IV, Lemma 3.3(1)). Therefore, our lifting problem has indeed a strict solution.

**Step 2.** We now solve the lifting problem

$$(***) \qquad \begin{array}{ccc} A \boxtimes \Delta^n & \xrightarrow{\ h\ } & X \\ {\scriptstyle j}\big\downarrow & \overset{l'}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \Delta^n \boxtimes \Delta^n & \xrightarrow[\ v\ ]{} & Y \end{array}$$

Consider the adjoint lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X_{*,n} \\ {\scriptstyle\gamma}\big\downarrow & \overset{l}{\nearrow} & \big\downarrow{\scriptstyle f_{*,n}} \\ \Delta^n & \longrightarrow & Y_{*,n} \end{array}$$

If $\gamma$ has strict LLP with respect to $f_{*,n} : X_{*,n} \to Y_{*,n}$, this problem has a strict solution. Hence, our original problem $(*)$ also has a strict solution, and we are done.

If $\gamma$ has weak LLP with respect to $f_{*,n} : X_{*,n} \to Y_{*,n}$, a lift $l : \Delta^n \to X_{*,n}$ exists, but the upper triangle commutes only up to a fiberwise homotopy $H : A \times \Delta^1 \to X_{*,n}$ (relative to $Y_{*,n}$). By adjunction, this gives rise to a lift

$$l' : \Delta^n \boxtimes \Delta^n \to X$$

in $(***)$. The upper triangle in $(***)$, however, is not, strictly speaking, homotopy commutative. Rather, instead of a homotopy we have a map $H' : (A \times \Delta^1) \boxtimes \Delta^n \to X$, the adjoint of $H$. Let $H''$ be the composition

$$H'' : d^*(A \times \Delta^1) \to (A \times \Delta^1) \boxtimes \Delta^n \xrightarrow{H'} X.$$

Here, the first map is adjoint to

$$(\mathrm{id}, \gamma) \times \mathrm{id}_{\Delta^1} : A \times \Delta^1 \to \mathrm{Diag}((A \times \Delta^1) \boxtimes \Delta^n) = (A \times \Delta^1) \times \Delta^n = A \times \Delta^n \times \Delta^1,$$

where $(\mathrm{id}, \gamma) : A \to A \times \Delta^n$ is the diagonal inclusion; see the paragraph before Lemma 11.6. It follows that the pair

$$l' : \Delta^n \boxtimes \Delta^n \to X, \quad H'' : d^*(A \times \Delta^1) \to X$$

is the desired solution to $(*)$. □

11.4. Sing **preserves fibrations.** We are finally ready to prove one of our main results.

**Theorem 11.8.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks that is a (weak) (trivial) Serre fibration and also a Reedy fibration. Then,* $\mathrm{Sing}(p) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ *is a (weak) (trivial) Kan fibration.*

*Proof.* Let $R_m(\mathcal{X}) := B(\mathcal{X})_{*,m}$ be the $m^{\text{th}}$ row of the bisimplicial set $B(\mathcal{X})$ (where $B(\mathcal{X})$ is defined in Definition 9.1). Note that we have

$$R_0(\mathcal{X}) = \mathrm{Ob}(\mathcal{X}_{\boldsymbol{\Delta}}), \; R_1(\mathcal{X}) = \mathrm{Mor}(\mathcal{X}_{\boldsymbol{\Delta}}), \; R_m(\mathcal{X}) = R_1(\mathcal{X}) \times_{R_0(\mathcal{X})} \times \cdots \times_{R_0(\mathcal{X})} R_1(\mathcal{X}).$$

It follows from Lemma 11.5 that, for every $m$, $B(p)_{*,m} : B(\mathcal{X})_{*,m} \to B(\mathcal{Y})_{*,m}$ is a (weak) (trivial) Kan fibration. Furthermore, $B(p)$ is a Reedy fibration of bisimplicial sets because, by assumption, $p_{\boldsymbol{\Delta}} : \mathcal{X}_{\boldsymbol{\Delta}} \to \mathcal{Y}_{\boldsymbol{\Delta}}$ is a Reedy fibration of simplicial groupoids, and the nerve functor $N : \mathsf{Gpd} \to \mathsf{sSet}$ preserves fibrations and limits (see the proof of Proposition 8.9). It follows now from Lemma 11.7 that $B(p)$ is a diagonal (weak) (trivial) Kan fibration. In other words, $\mathrm{Sing}(p) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ is a (weak) (trivial) Kan fibration. □

**Corollary 11.9.** *Let $\mathcal{X}$ be a Reedy fibrant Serre topological stack. Then,* $\mathrm{Sing}(\mathcal{X})$ *is a Kan simplicial set.*

**Corollary 11.10.** *For every (weak) (trivial) Serre fibration of Serre stacks $p : \mathcal{X} \to \mathcal{Y}$, there exists a strictly commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;p\;} & \mathcal{Y} \\ {\scriptstyle\sim}\big\downarrow{\scriptstyle g} & \nearrow_{p'} & \\ \mathcal{X}' & & \end{array}$$

*where $p'$ is a (weak) (trivial) Serre fibration as well as an injective (hence, also Reedy) fibration, and $g : \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ is an equivalence of Serre stacks. In particular,* $\mathrm{Sing}(p') : \mathrm{Sing}(\mathcal{X}') \to \mathrm{Sing}(\mathcal{Y})$ *is a (weak) (trivial) Kan fibration.*

*Proof.* This follows from Proposition 8.14 and Theorem 11.8. (Also see Remark 7.5.) □

**Corollary 11.11.** *For every Serre stack $\mathcal{X}$ there exists a Serre stack $\mathcal{X}' \sim \mathcal{X}$ equivalent to it that is Reedy fibrant (hence,* $\mathrm{Sing}(\mathcal{X}')$ *is a Kan simplicial set).*

## 12. Singular functor preserves weak equivalences

In this section, we prove that the singular functor has the correct homotopy type by showing that it takes a weak equivalence of topological stacks to a weak equivalence of simplicial sets (Theorem 12.2). We begin with a special case.

**Proposition 12.1.** *Let $\mathcal{X}$ be a Serre stack, and let $\varphi : X \to \mathcal{X}$ be a trivial weak Serre fibration with $X$ (equivalent to) a topological space (i.e., $X$ is a classifying space for $\mathcal{X}$ in the sense of Theorem 3.4). Then,* $\mathrm{Sing}(\varphi) : \mathrm{Sing}(X) \to \mathrm{Sing}(\mathcal{X})$ *is a weak equivalence of simplicial sets.*

*Proof.* We may assume that $\mathcal{X}$ is Reedy fibrant (Corollary 9.4 and Corollary 11.11). By Corollary 9.4 and Corollary 11.10, we may assume that $\varphi : X \to \mathcal{X}$ is a trivial weak Serre fibration as well as a Reedy fibration. Note that we are not insisting on $X$ being *isomorphic* to but only equivalent to a topological space $X'$.

Observe that we can always find a pair of inverse equivalences between $X$ and $X'$. On the one hand, we have that $\pi_0(X(T)) = X'(T)$ for every $T \in \mathsf{Top}$, so we have an equivalence $p : X \to X'$. In particular, $X(X') \to X'(X')$ is an equivalence of groupoids (the latter is actually a set). Picking $f \in X(X')$ in the inverse image of $\mathrm{id} \in X'(X')$ and applying Yoneda's lemma, we find the desired inverse $f : X' \to X$ to $p$.

Now, by Theorem 11.8, $\mathrm{Sing}(\varphi) : \mathrm{Sing}(X) \to \mathrm{Sing}(\mathcal{X})$ is a trivial weak Kan fibration, and $\mathrm{Sing}(\mathcal{X})$ is Kan. Furthermore, the conditions of Lemma 11.4 are satisfied as the map $\mathrm{Sing}(X') \to \mathrm{Sing}(X)$ is a weak equivalence (Corollary 9.4) and $\mathrm{Sing}(X')$ is Kan. So, by Lemma 11.4, $\mathrm{Sing}(\varphi)$ is a weak equivalence. □

**Theorem 12.2.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a weak equivalence of Serre stacks. Then,* $\mathrm{Sing}(f) : \mathrm{Sing}(\mathcal{X}) \to \mathrm{Sing}(\mathcal{Y})$ *is a weak equivalence of simplicial sets.*

*Proof.* We can choose classifying atlases $\varphi : X \to \mathcal{X}$ and $\psi : Y \to \mathcal{Y}$ (in the sense of Theorem 3.4) fitting in a 2-commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ {\scriptstyle\varphi}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle\psi} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y} \end{array}$$

This is done as follows. Choose classifying atlases $\psi : Y \to \mathcal{Y}$ and $h : X \to \mathcal{X}\tilde{\times}_{\mathcal{Y}} Y$. Set $\varphi = \mathrm{pr}_1 \circ h$ and $f' = \mathrm{pr}_2 \circ h$; by ([No14], Lemma 3.8), $\varphi$ is again a trivial weak Serre fibration.

Now, by the two-out-of-three property, $f'$ is a weak equivalence. Applying Sing, we find a homotopy commutative diagram in simplicial sets where $\mathrm{Sing}(f')$, $\mathrm{Sing}(\varphi)$ and $\mathrm{Sing}(\psi)$ are weak equivalences of simplicial sets (by Proposition 12.1, also see Remark 9.2). Therefore, $\mathrm{Sing}(f)$ is also a weak equivalence by the two-out-of-three property. □

**Corollary 12.3.** *Let $\mathcal{X}$ be a Serre topological stack, and let $\mathbb{X} = [R \rightrightarrows X]$ be a groupoid presentation for it. Then, there is a natural weak equivalence*

$$\mathrm{Sing}(\|N(\mathbb{X})\|) \to \mathrm{Sing}(\mathcal{X}),$$

*of simplicial sets, where the left-hand occurrence of* Sing *is the classical singular chains functor, and $\| - \|$ denotes the fat geometric realization.*

*Proof.* This follows from the fact that there is a natural map $\|N(\mathbb{X})\| \to \mathcal{X}$, and this map is a classifying space for $\mathcal{X}$; see [No14], Corollary 3.17 and [No12], Theorem 6.3. □

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