Eigenvalue curves of tridiagonal random matrices

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$n \times n$ Jacobi matrices:

$$A_n = \begin{pmatrix} q_1 & b_1 & & & \\ a_1 & q_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-2} & q_{n-1} & b_{n-1} \\ & & & a_{n-1} & q_n \end{pmatrix}$$

Eigv.eq.: $\quad b_k\psi_{k+1}+q_k\psi_k+a_{k-1}\psi_{k-1}=z\psi_k,$

$\psi_0 = 0, \psi_{n+1} = 0$ (bound.conds.)

'Circulant' Jacobi matrices:

$$J_n = \begin{pmatrix} q_1 & b_1 & & & a_0 \\ a_1 & q_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-2} & q_{n-1} & b_{n-1} \\ b_n & & & a_{n-1} & q_n \end{pmatrix}$$

Eigv.eq.: $\quad b_k\psi_{k+1}+q_k\psi_k+a_{k-1}\psi_{k-1}=z\psi_k,$

$\psi_0 = \psi_n, \psi_1 = \psi_{n+1}$ (bound.conds.)

Non-Hermitian Jacobi matrices - motivation.

Two kinds of random Jacobi matrices:

(1) positive a_k and b_k, and real q_k

(2) the sign (phase if complex) of a_k and b_k is chosen randomly.

Eigenvalue counting measure

If A_n is an $n \times n$ matrix with eigenvalues $z_1, \ldots, z_n$ then

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j}$$

is the eigenvalue counting measure for A_n,

$$\int_K d\mu_n = \frac{\#\{\text{eigvs. of } A_n \text{ in } K\}}{n}$$

Given a sequence $\{A_n\}_n$ of matrices of increasing dimension n, does μ_n converge when $n \to \infty$?

$$\int_{\mathbb{C}} f(z) d\mu_n \to \int_{\mathbb{C}} f d\mu?$$

Describe the limiting measure μ?

Tools:

· moments: $\int z^k d\mu_n = \frac{1}{n} \sum_j z_j^k = \frac{1}{n} \operatorname{tr} A_n^k$

· resolvents: $g_n(u) = \int \frac{d\mu_n(z)}{z-u} = \frac{1}{n} \sum\limits_j \frac{1}{z_j - u} = \frac{1}{n} \operatorname{tr}(A_n - uI_n)^{-1}$

$g_n(u)$ is the Stieltjes (Cauchy) transform of μ_n. Stieltjes-Perron inversion formula!

· determinants:

$p_n(u) = \int \ln|z-u| d\mu_n = \frac{1}{n} \sum\limits_j \ln|z_j - u| = \frac{1}{n} \ln|\det(A_n - uI_n)|$

$g_n(u)$ is the log-potential of μ_n, $\frac{1}{2\pi}\Delta p_n = \mu_n$ as distributions (Poisson's equation)

If Δ is the distributional Laplacian in $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ then

$$\frac{1}{2\pi}\Delta \ln|z-a| = \delta_a$$

as distributions (Gauss-Green), hence

$$\mu_n = \frac{1}{2\pi n}\sum_j \Delta \ln|z-z_j| = \frac{1}{2\pi}\Delta p_n$$

where $p_n(z) = \frac{1}{n}\ln|\det(zI_n - A_n)| = \ln|\det(zI_n - A_n)|^{1/n}$.

Thm.1 (Widom 1993, Goldshed-K. 2003). Suppose:

(i) $p_n(z) \to p(z)$ as $n \to \infty$, a.e. in $\mathbb{C}$

(ii) $M_n := \prod_{|z_j|\geq 1} |z_j| \leq e^{nC}$ for all n large.

Then p is loc. integrable, $\mu := \frac{1}{2\pi}\Delta p$ is a unit mass measure and μ_n converges weakly to μ as $n \to \infty$. Also

$$\int_{|z|\geq 1} \ln|z|\, d\mu \leq C.$$

Note: (Weyl's inequalities)

$$M_n^2 = \prod_{|z_j|\geq 1} |z_j|^2 \leq \prod_{j=1}^{\kappa} s_j^2 \leq \prod_{j=1}^{n}(s_j^2+1) = \det(A_nA_n^* + I_n).$$

Type II Jacobi matrices (no corner entries and a_j, q_j and b_j are random complex numbers).

Let f_k be the solution of the eq.

$$b_k f_{k+1} + q_k f_k + a_{k-1} f_{k-1} = z f_k, \qquad k = 1, 2, \ldots$$

with the initial data $f_0 = 0$ and $f_1 = 1$.

f_{n+1} is a polyn. in z of deg. n and $f_{n+1}(z_j) = 0$ all j. Hence

$$f_{n+1}(z) = \kappa_n \prod_{j=1}^{n} (z - z_j) = \kappa_n \det(zI_n - A_n),$$

$\kappa_n = \prod_{j=1}^{n} 1/b_j$.

Lyapunov exponent:

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{2n} \ln[|f_{n+1}(z)|^2 + |f_n(z)|^2]$$

$\gamma(z)$ is non-random and the limit exists with prob. one.

Assume (a_j, q_j, b_j) is a sequence of i.i.d. vectors in $\mathbb{C}^3$ such that $E(|a_j|^{\pm\delta})$, $E(|b_j|^{\pm\delta})$ and $E(|q_j|^{\delta})$ are all finite for some $\delta > 0$.

Thm.2 With the above assumptions:

(a) With prob. one, $\mu_n \Rightarrow \mu = \frac{1}{2\pi}\Delta\gamma$ as $n \to \infty$.

(b) (Thouless Formula)

$$\gamma(z) = \int_{\mathbb{C}} \ln|z - w| d\mu(w) - E(\ln|b_1|)$$

(c) μ is log-Hölder continuous.

Type I Jacobi matrices (circulant)

$$J_n(t) = \begin{pmatrix} q_1 & t & & & t^{-1} \\ t^{-1} & q_2 & t & & \\ & \ddots & \ddots & \ddots & \\ & & t^{-1} & q_{n-1} & t \\ t & & & t^{-1} & q_n \end{pmatrix}$$

where q_k are real and $t \geq 1$. This is as before but now $a_k = t^{-1}$, $b_k = t$ for all k. (Lose nothing of interest.)

Two observations:

(A) $J_n(t) = DH_n(t^n)D^{-1}$, with $D = \mathrm{diag}(1, t, \ldots, t^n)$ and

$$H_n(t^n) = \begin{pmatrix} q_1 & 1 & & & t^{-n} \\ 1 & q_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & q_{n-1} & 1 \\ t^n & & & 1 & q_n \end{pmatrix}$$

Note: now asymmetry is due to the b.c. only, have finite rank perturbation of a symmetric matrix.

(B) For any $t \neq 0$:

$$\det[zI_n - H_n(t^n)] = \det[zI_n - H_n(i)] - (t^n + t^{-n}).$$

$$\Downarrow$$

$\therefore$ the eigenvalues of $J_n(t)$ are determined by

$$\det[zI_n - H_n(i)] = t^n + t^{-n}.$$

Note: $H_n(i)$ is Hermitian.

Log-potential of the eigenvalue distr. of $J_n(t)$:

$$p_n(z)=\frac{1}{n}\ln|\det[z-J_n(t)]|=\frac{1}{n}\ln|\det[z-H_n(t^n)]|.$$

Recall the determinantal relation:

$$\det[zI_n-H_n(t^n)]=\det[zI_n-H_n(i)]-(t^n+t^{-n}).$$

Note: $t^n+t^{-n}=e^{n\ln t+o(n)}$ and $|\det[zI_n-H_n(i)]|=e^{nu_n(z)}$ where

$$u_n(z)=\frac{1}{n}\sum_j \ln|z-\lambda_j|=\int_{\mathbb{R}}\ln|z-\lambda|dN_n(\lambda).$$

Here $N_n(\lambda)$ is the eigenvalue counting measure for the Hermitian matrix $H_n(i)$.

Assume q_k are independent identically distributed random variables such that $E(\ln(1+|q_k|)$ is finite. Then with probability one $dN_n(\lambda)$ converges to non-random $dN(\lambda)$ and, for every non-real z,

$$u_n(z)\to u(z)=\int_{\mathbb{R}}\ln|z-\lambda|dN(\lambda).$$

Hence for any non-real z

$$|\det[zI_n-H_n(i)]|=e^{nu(z)+o(n)},\qquad \operatorname{Im} z\neq 0$$

and

$$p_n(z)\to p(z)=\begin{cases} u(z) & \text{if } u(z)>\log t\\ \log t & \text{if } u(z)<\log t\end{cases}$$

By the way of Thm. 1,

Thm.3 If μ_n is the eigenvalue counting measure for the random circulant Jacobi matrix $J_n(t)$ then with probability one μ_n converges weakly to $d\mu=\frac{1}{2\pi}\Delta p$ in the limit $n\to\infty$.

Eigenvalue curves

Limiting eigenvalue distribution μ is supported by
curve $\mathcal{L} = \{z \in \mathbb{C} : \ u(z) = \ln t\}$
intervals $I = \{\lambda \in \text{supp}\, dN \subseteq \mathbb{R} : u(\lambda + i0) > \ln t\}$.

On $\mathcal{L}$: $d\mu = \rho(z)dl$, where dl is the arc-length of $\mathcal{L}$, with density $\rho(z) = \frac{1}{2\pi}\left|\int \frac{dN(\lambda)}{\lambda - z}\right|$.

On I: $d\mu = dN$, i.e. $\mu([a,b]) = N(b) - N(a)$ for any interval $[a,b] \subset I$.

Define $\Lambda = \{x \in \mathbb{R} : \int \ln|x-\lambda| dN(\lambda) < \ln|t|\}$

t_c: $\qquad \ln t_c = \inf_{x\in\mathbb{R}} \int \ln|x-\lambda| dN(\lambda)$

Note: $t_c > 1$ (specific to random q_k, Thouless + Furstenberg), and $\mathcal{L} = \emptyset$ if $1 \le t \le t_c$!

Consider $t > t_c$. Then $\Lambda = \cup_j(\alpha_j, \beta_j)$, and the curve $\mathcal{L}$ consists of closed branches

$$y = \pm y_j(x) \quad \alpha_j \le x \le \beta_j$$

where $y_j(x)$ for $\alpha_j < x < \beta_j$ is the positive solution of

$$\int \ln|x + iy - \lambda| dN(\lambda) = \ln|t|.$$

Recall that the eigenvalues of $J_n(t)$ are determined by

$$\det[zI_n - H_n(i)] = t^n + t^{-n}.$$

The polynomial $P_n(z) = \det[zI_n - H_n(i)] = \prod(z - \lambda_j)$ has real roots ($H_n(i)$ is Hermitian!) and we know the limiting distribution of these roots ($dN(\lambda)$!)

Problem: find the distribution of roots of eq. $P_n(z) = t^n + t^{-n}$ for large n, where $P_n(z)$ is a polynomial of degree n with real zeros.

Finite-n picture

Denote $g_n = t^n + t^{-n}$. Roots of $P_n(z) = g_n$ lie on lemniscate

$$\mathcal{L}_n : \qquad |z - \lambda_1| \cdot \ldots \cdot |z - \lambda_n| = |g_n|$$

consisting of k symmetric 'ovals', $k \le n$.

On each 'oval':

$$P_n(z) = g_n \;\; \Leftrightarrow \;\; \arg P_n(z) = \arg g_n \pmod{2\pi}$$

and the no. of roots is exactly the same as the no. of λ_j in the interior of the 'oval'.

Now let $n \to \infty$. How $\mathcal{L}_n$ evolves?

We know that $dN_n(\lambda) \Rightarrow dN(\lambda)$, therefore

$$\frac{1}{n} \ln P_n(z) \to \int_{\mathbb{C}} \ln(z - \lambda) dN(\lambda)$$

uniformly on compact sets off $\mathbb{R}$. It follows from this that with prob. one, part of $\mathcal{L}_n$ collapses onto the real axis, the remaining part tends to the curve

$$\ln t = \int_{\mathbb{C}} \ln |z - \lambda| dN(\lambda).$$

Fix j, $\varepsilon > 0$, and consider the vertical strip $\alpha_j + \varepsilon \leq x \equiv \mathrm{Re}\, z \leq \beta_j - \varepsilon$.

Thm.4 With probability one, the lemniscate $\mathcal{L}_n = \{z \in \mathbb{C} : |\det[zI_n - H_n(i)]| = t^n + t^{-n}\}$ is represented in this strip by two analytic arcs

$$y = \pm y_{j,n}(x) \quad \alpha_j + \varepsilon \leq x \leq \beta_j - \varepsilon,$$

provided $n > n_1(\varepsilon, q, t)$, and

$$\lim_{n\to\infty} y_{j,n}(x) \;\; = \;\; y_j(x)$$

uniformly in $x \in [\alpha_j + \varepsilon, \beta_j - \varepsilon]$.

Consider the arc $A_{j,n}:\ y = y_{j,n}(x) \quad \alpha_j + \varepsilon \leq x \leq \beta_j - \varepsilon$. Label the eigvs. of $J_n(t)$ on $A_{j,n}$ as follows

$$z_l = x_l + iy_{j,n}(x_l) \text{ where } x_1 < x_2 < ... < x_m.$$

Thm.5 With probability one,

$$n(z_{l+1} - z_l) = \frac{i}{r(z_l)} + \delta_n(z_l, z_{l+1})$$

where $r(z) = \frac{1}{2\pi} \int \frac{dN(\lambda)}{\lambda - z}$ and

$$\lim_{n\to\infty} \delta_n(z_l, z_{l+1}) = 0$$

uniformly in $z_l, z_{l+1} \in [\alpha_j + \varepsilon, \beta_j - \varepsilon]$.