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MAS200 — Actuarial Statistics

Facts from Probability: — Spring 2002

Notation:

$P(A)$ = Probability that the event described by A occurs.
$P(A|B)$ = Probability that A occurs given that B has occurred (conditional probability);

$A \cap B$ = A and B.
$A \cup B$ = A or B.

Note:

(a) $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$.

(b) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$, always.

(c) If A and B are mutually exclusive, i.e. $A\cap B = \emptyset$, then $P(A\cup B) = P(A) + P(B)$.

(d) If $A_1 \cap A_2 = \emptyset$ then $P(A_1\cup A_2|B) = P(A_1|B) + P(A_2|B)$.

(e) If $A \subset B$ (i.e. A implies B) then $P(A|B) = \dfrac{P(A)}{P(B)}$.

(f) $P(A\cap B) = P(A)P(B)$ **if and only if** the events A and B are independent.

Cumulative *distribution function* (c.d.f.): $F_X(x) = P(X \le x)$

Two types of random variables:- discrete and continuous

Discrete random variables:

X takes on values from a discrete set $\{x_1, x_2, \ldots\}$. In this case $F_X(x) = \sum_{x_k\le x} P(X = x_k)$.

Continuous random varaibles X:

There exists a continuous function $f_X(x)$, called *probability density function* (p.d.f.), such that

(g) $F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)du$.

The c.d.f. $F_X(x)$ and p.d.f. $f_X(x)$ for a continuous r.v. X are related via $\frac{d}{dx}F_X(x) = f_X(x)$ which holds at every x where the derivative exists.

Note:

(h)
$$\begin{aligned} P(X > x) &= 1 - F_X(x);, && \text{(holds always, follows from } P(X>x)+P(X\le x)=1) \\ &= \sum_{x_k > x} P(X = x_k) && \text{(if } X \text{ is a discrete r.v.)} \\ &= \int_{x}^{+\infty} f_X(u)du. && \text{(if } X \text{ is a continuous r.v.)} \end{aligned}$$

(i)
$$\begin{aligned} P(x < X \le y) &= F_X(y) - F_X(x); && \text{(holds always)} \\ &= \sum_{x < x_k \le y} P(X = x_k) && \text{(if } X \text{ is a discrete r.v.)} \\ &= \int_{x}^{y} f_X(u)du. && \text{(if } X \text{ is a continuous r.v.)} \end{aligned}$$

In particular, for continuous X:

$$\begin{aligned} P(x \le X \le x+h) &= F_X(x+h) - F_X(x) \\ &\simeq \frac{d}{dx}F_X(x)\times h, \quad \text{for small } h, \\ &= f_X(x)h \end{aligned}$$

(j) If X is a continuous random variable then

$$P(x_1 < X \le x_2) = P(x_1 \le X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2).$$

Expected Value of X:

$$\begin{aligned} E(X) &= \sum_{x_k} x_k P(X = x_k), && \text{if } X \text{ is a discrete random variable;} \\ &= \int_{-\infty}^{+\infty} u\, f_X(u)\, du, && \text{if } X \text{ is a continuous random variable.} \end{aligned}$$

Note: $E(aX + bY) = aE(X) + bE(Y)$.

Variance of X (mean quadratic deviation from $E(X)$):

$$\mathrm{var}(X) = E([X - E(X)]^2) = E(X^2) - [E(X)]^2$$

Note: $\mathrm{var}(aX + bY) = a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y) - ab\,\mathrm{cov}(X,Y)$, where $\mathrm{cov}(X,Y) = E([X - E(X)][Y - E(Y)])$ is the covariance of X and Y. If X and Y are independent then $\mathrm{cov}(X,Y) = 0$ and $\mathrm{var}(aX + bY) = a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y)$.

Conditioning a random variable X by the event $X > t$.

If X is a continuous random variable and t belongs to the range of its values, we can define a new random variable $(X-t)|(X>t)$ (reads $X-t$ given $X>t$). The probability distribution of this new random variable is the probability distribution of $X-t$ conditioned by the event $X>t$, i.e.

$$\begin{aligned} F_{(X-t)|(X>t)}(s) &\stackrel{\text{def.}}{=} P(X-t\le s|X>t) \\ &= P(t<X\le t+s|X>t) \end{aligned}$$

This new random variable is obviously non-negative and its p.d.f. $f_{(X-t)|(X>t)}$ can be obtained by calculating $P(s<X-t\le s+h \mid X>t)$ for small h and positive s:

$$\begin{aligned} f_{(X-t)|(X>t)}(s)\times h &\doteq P(s<(X-t)\le s+h \mid X>t) && \text{[by (i)]} \\ &= \frac{P(s<X-t\le s+h)}{P(X>t)} && \text{[by (a) and (e)]} \\ &= \frac{P(s+t<X\le s+t+h)}{P(X>t)} \\ &\doteq \frac{f_X(s+t)\times h}{P(X>t)}, && \text{[by (i)]} \end{aligned}$$

Therefore

(k) $$f_{(X-t)|(X>t)}(s) = \frac{f_X(s+t)}{P(X>t)} = \frac{f_X(s+t)}{1-F_X(t)} \text{ if } s\ge 0 \text{ and } f_{(X-t)|(X>t)}(s)=0 \text{ if } s<0.$$

The conditional expectation of $X-t$ given $X>t$ is denoted by $E(X-t|X>t)$ and

(l) $$E(X-t|X>t) = \int_0^\infty s\, f_{(X-t)|(X>t)}(s)\,ds = \frac{\int_0^\infty s\, f_X(s+t)\,ds}{P(X>t)} = \frac{\int_t^\infty (u-t)\, f_X(u)\,du}{P(X>t)}.$$

Examples of distributions:

1. Bernoulli distribution (discrete-type): $X\sim \text{Bernoulli}(p)$,

   X takes on either 1 or 0 (success or failure); $P(X=1)=p$, $P(X=0)=q$; $p+q=1$; $E(X)=p$, $\text{var}(X)=pq$

2. Binomial distribution (discrete-type): $X\sim \text{Bin}(n,p)$,

   X takes on the values $0,1,2,\ldots n$; $P(X=k)=C_k^n p^k q^{n-k}$, $k=0,1,\ldots,n$; $p+q=1$; $E(X)=np$, $\text{var}(X)=npq$

   If $X_1,X_2,\ldots X_n$ are mutually independent and $X_j\sim\text{Bernoulli}(p)$ for all j then

   $$X_1+X_2+\ldots X_n\sim \text{Bin}(n,p).$$

3. Uniform distribution on $[0,1]$ (continuous-type): $X\sim\text{Uniform}[0,1]$,

   X can take on any value between 0 and 1; $P(a\le X\le b)=b-a$ for any $0\le a<b\le 1$.
   p.d.f.: $f_X(x)=1$ if $x\in[0,1]$ and $f_X(x)=0$ otherwise.
   $E(X)=\frac{1}{2}$, $\text{var}(X)=\frac{1}{3}$

4. Exponential distribution (continuous-type): $X\sim\text{Exp}(\lambda)$,

   X can take on any non-negative value;
   p.d.f.: $f_X(x)=\lambda e^{-\lambda x}$ if $x\ge 0$ and $f_X(x)=0$ otherwise
   c.d.f.: $F_X(x)=1-e^{-\lambda x}$ if $x\ge 0$ and $F_X(x)=0$ otherwise
   $E(X)=\frac{1}{\lambda}$, $\text{var}(X)=\frac{1}{\lambda^2}$

   If $X\sim\text{Exp}(\lambda)$ then $(X-t)|(X>t)\sim\text{Exp}(\lambda)$ as well:

   $$\begin{aligned} f_{(X-t)|(X>t)}(s) &= \frac{f_X(s+t)}{1-F_X(t)} = \frac{\lambda e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= \lambda e^{-\lambda s} = f_X(s) && s,t\ge 0. \end{aligned}$$

   Also, if $X\sim\text{Exp}(\lambda)$ then $P(X-t>s|X>t)=P(X>s)$.