Queen Mary, University of London

MAS200 ACTUARIAL STATISTICS (B.Sc. examination by course units)

Duration: 2 hours
Date and time: 22 May 2001, 10:00 – 12:00

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best THREE questions will be counted.

You are permitted to use an electronic calculator in this examination but you may not use any preprogrammed or graphical features which it may have. Please state on your answer book the name and type of machine used.

Question 1. (*30 marks*) Give your answers in parts (b), (d) - (f) to 2 decimal places.

(a) (*3 marks*) Define what is meant by the force of interest $\delta(t)$.

In the remaining parts of this question the force of interest is assumed to be constant, i.e. $\delta(t) = \delta$ at any time t, time being measured in years.

(b) (*4 marks*) $\delta = 0.15$. What is the accumulated value, at time $t > 0$, of one unit of money invested at time $t = 0$? Find the corresponding annual rate of interest (growth).

(c) (*7 marks*) Define what is meant by $i^{(p)}$, the nominal rate of interest per annum compounded $p$thly. Prove that $\lim_{p\to\infty} i^{(p)} = \delta$.

(d) (*5 marks*) State the relation between the nominal rate of interest compounded quarterly and the corresponding effective annual rate of interest.

A bank credits interest on deposits quarterly at rate 1.5% *per quarter*. What is the nominal interest rate? What is the corresponding effective annual rate of interest?

(e) (*5 marks*) Mr. Smith is under an obligation to make a single payment of £1000 to a building society in one year's time from now. He offers to discharge his liability for this debt by making an appropriate single payment in one month's time from now. On the basis of a constant rate of interest of 6% per annum effectively, find the appropriate payment if Mr. Smith's offer is accepted by the building society.

(f) (*6 marks*) A washing machine can be purchased for £329 cash or by 24 equal monthly repayments, the first one being due at the moment of purchase. Find the amount of each repayment if the interest charged on this transaction is 27% per annum effectively, i.e. APR=27%.

Turn over ...

Question 2. (*30 marks*)

(a) (*4 marks*) Define the survival function $s(x)$ in terms of the age-at-death random variable X. If $s(50) = 0$, what is the value of $s(x)$ at age $x = 51$?

(b) (*5 marks*) Define in words what is meant by ${}_tq_x$. Express ${}_tq_x$ in terms of X and in terms of $T(x)$, the future lifetime at age x. Obtain ${}_tq_x$ in terms of the survival function.

(c) (*7 marks*) Define what is meant by the force of mortality $\mu(x)$ and explain, as fully as you can, why $\mu(x)$ can be interpreted as the expected rate of instantaneous deaths at age x.

(d) In a population of rats, no rat can survive 5 complete years and the probability that a newborn rat will survive to age x is $\frac{1}{5}(5-x)$ for all $0 \le x \le 5$.

(i) (*2 marks*) What is the probability that a two-year-old rat will survive the following year?

(ii) (*6 marks*) Determine the probability mass function for $K(2)$, the number of complete years to be lived in the future by a two-year-old rat.

(e) (*6 marks*) The inhabitants of a remote island do not live for more than 80 years. Their law of mortality is such that at age x, $60 \le x \le 80$, the force of mortality is $\dfrac{1}{80-x}$. What is the complete expectation of further life at the age of 60 on this island?

Question 3. (*30 marks*) Assume mortality according to English Life Table No. 12 – Males. The table is appended to the end of this examination paper. Give your answers in (a)-(c) and (e)-(g) to 2 decimal places.

(a) (*3 marks*) What is the probability that a man aged 30 will die during the following 10 years?

(b) (*3 marks*) What is the probability that a man aged 30 will survive to the age of 40 and die before attaining the age of 60?

(c) (*3 marks*) What is the probability that a newborn boy will survive to the age of 40 and die before attaining the age of 60?

(d) (*3 marks*) Consider a group of 100 men, each of whom are age 30 now. What is the expected number of men from this group who will die when they are at least 60 but are not yet 62? Round your answer to the nearest integer.

(e) (*7 marks*) Mr. Smith retires through ill-health at the age of 60. With the exception of a select period of one year following his retirement, Mr. Smith's future mortality is described by English Life Table 12 - Males. The probability that Mr. Smith will survive the year following his retirement is 0.9.

Find the *curtate* expectation of Mr. Smith's further life on his retirement, i.e. find $e_{[60]}$.

(f) (*4 marks*) Consider a pure endowment policy for a five-year-old boy which is being effected on his fifth birthday. Under this policy, the boy will receive £5,000 on his 21st birthday if he survives, and no money will be paid out if he dies before attaining the age of 21. Assuming interest at rate 13% per annum effectively, calculate the expectation and standard deviation of the present value of the benefit payment under this policy.

(g) (*7 marks*) A life company withdraws benefit payments from an investment fund earning interest at rate 13% per annum effectively. Suppose that the company has just sold a block of 100 pure endowment policies identical to the one described in (f). Using the results obtained in (f), and a normal approximation for the total present value of the benefit payments, calculate the minimum amount to be invested by the company at the present time so that the probability is approximately 0.95 that sufficient funds will be on hand to withdraw the benefit payment for each boy.

(If a random variable N has the standard normal distribution than $P(N < 1.645) = 0.95$.)

Question 4. (*30 marks*)

Answer (a) – (d) on the basis of table A1967–70 with 4% interest, appended to the end of this examination paper. Use *ultimate* (non-select) values.

(a) (*2 marks*) Consider a whole-life annuity-due, issued to a life aged 65, which pays £1 annually. Calculate the expected present value of this annuity to 3 decimal places.

(b) (*4 marks*) Mr. Baker, on his retirement at the age of 65, purchases a whole-life annuity with a lump sum of £20,000. The annuity is such that it will pay annually a fixed amount to Mr. Baker whilst he is alive, with the first payment being due on Mr. Baker's 66th birthday. Ignoring expenses, how much will Mr. Baker receive annually? Give your answer to 2 decimal places.

(c) (*3 marks*) A whole-life assurance with a sum assured of £30,000 is being issued to a life aged 65. The sum assured is payable at the end of the year of death of the life assured. Find the expected present value of the benefit payment. Give your answers to 3 decimal places.

(d) (*3 marks*) Premiums for the whole-life assurance in (c) are paid annually in advance for life. State the equivalence principle and use it to calculate the annual premium (net value) to 2 decimal places.

(e) (*5 marks*) Consider a whole-life annuity-due of 1 per annum payable $p$thly to a life aged x at the moment of purchase. Its expected present value (at the moment of purchase) is denoted by the symbol $\ddot{a}_x^{(p)}$. Show that

$$\ddot{a}_x^{(p)} = \frac{1}{p} \sum_{k=0}^{\infty} \frac{D_{x+\frac{k}{p}}}{D_x}, \quad \text{where } D_x = v^x l_x.$$

Turn over ...

(f) (*6 marks*) Prove that $\ddot{a}_x^{(p)} \leq \frac{1}{d^{(p)}}$ for all $x \geq 0$, where $d^{(p)}$ is the nominal rate of discount convertible $p$thly.

(g) (*7 marks*) Use linear interpolation on D_{x+t} and the result in (e) to obtain the approximate relation $\ddot{a}_x^{(p)} \simeq \ddot{a}_x - \frac{p-1}{2p}$

Question 5. (*30 marks*)

Consider females in a growing population. Let $n_x(t)$ be the expected number of females aged x in the population at time t. Let $p_x(t)$ be the probability of surviving to age $(x+1)$ for a female aged x at time t. Also let $b_x(t)$ be the expected number of female births in the year t to $t+1$ to a female aged x at time t.

(a) (*12 marks*) Obtain equations expressing $n_x(t+1)$ in terms of $n_x(t)$ for all x and t.

(b) Let $b_x(t)$ be independent of t, so that $b_x(t) = b_x$ for all t.

  (i) (*7 marks*) If the expected numbers for the different ages stay constant over time, i.e. if $n_x(t) = n_x$ for all t, show that $p_x(t)$ is independent of t for all x and that the expected number of daughters during the lifetime of each female is 1.

  (ii) (*11 marks*) If $n_x(t) = \lambda^t n_x$ for some positive λ, show that the probabilities $p_x(t)$ and the survival function $s(x)$ are constant over time. For given values of b_x, show that λ is a solution to a certain equation and that, in the interval $0 < \lambda < +\infty$, this equation has a unique root.

End of examination questions. A list of selected formulae and Tables ELT-12 and A1967-70 follow.

Present values of annuities-certain:

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{1-v} \qquad a_{\overline{n}|} = v\,\ddot{a}_{\overline{n}|}$$

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{1}{p}\left[\frac{1-v^n}{1-v^{\frac{1}{p}}}\right] \qquad a^{(p)}_{\overline{n}|} = v^{\frac{1}{p}}\,\ddot{a}^{(p)}_{\overline{n}|}.$$

Expected present values of life annuities:

$$\ddot{a}_x = \frac{N_x}{D_x} \qquad \ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x} \qquad a_x = \ddot{a}_x - 1$$

$$\ddot{a}^{(p)}_x \simeq \ddot{a}_x - \frac{p-1}{2p} \qquad a^{(p)}_x \simeq a_x + \frac{p-1}{2p} \qquad a^{(p)}_x = \ddot{a}^{(p)}_x - \frac{1}{p}$$

Conversion relationships:

$$\bar{A}_x = 1 - \delta\bar{a}_x$$

$$A_x = 1 - d\,\ddot{a}_x$$

$$A^{(p)}_x = 1 - d^{(p)}\,\ddot{a}^{(p)}_x$$

$$A^{(p)}_{x:\overline{n}|} = 1 - d^{(p)}\,\ddot{a}^{(p)}_{x:\overline{n}|}$$

Turn over ...