On absolute moments of characteristic polynomails of a class of complex random matrices

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Spectral determinants: $|\det(zI-W)|^2 = \det(zI-W)(zI-W)^*$.

Keating-Snaith conjecture: $\langle|\det(I{-}zU)|^s\rangle_{U(n)}$ predicts moments of $\zeta(1/2{+}it)$

Averages of products and ratios of characteristic polynomials for random Hermitian and unitary matrices - extensively studied during last decade.

Our motivation is different: distribution of complex eigenvalues.

Why moments of $|\det(zI-W)|^2$ are relevant?

Poisson equation: $d\mu_n = \frac{1}{2\pi}\Delta p_n$;

$d\mu_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{z_j}$ eigenvalue counting measure;

$p_n(z) = \int \log|z-a|d\mu_n(a)$ its log-potential, $z = x+iy$;

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ distributional Laplacian

Under appropriate conditions, if $p_n \to p$ then $d\mu_n \Rightarrow \frac{1}{2\pi}\Delta p$.

Note: $p_n(z) = \frac{1}{2n}\log|\det(zI-W)|^2$, and

$G(s) = \langle|\det(zI{-}W)|^{2s}\rangle_W$ is a generating fnc for $\langle\log|\det(zI{-}W)|^2\rangle_W$.

[We can handle integer moments only: $G(m)$ for integer m.]

For some RM ensembles, the mean eigv. density is prop to $\langle|\det(zI-W)|^2\rangle_W$.

Two examples

1. Gaussian matrices W_n of dimension n (Edelman, Edelman-Kostlan-Shub):

Mean density of eigenvalues of complex W_n:

$$\rho_n(x,y) = Const\, e^{-|z|^2} \left\langle |\det(zI_{n-1} - W_{n-1})|^2 \right\rangle_{W_{n-1}}$$

Mean density of complex eigenvalues of real W_n:

$$\rho_n(x,y) = Const\, y e^{-(x^2-y^2)}\, \mathrm{erfc}(y) \left\langle |\det(zI_{n-2} - W_{n-2})|^2 \right\rangle_{W_{n-2}}$$

Mean density of real eigenvalues of real W_n:

$$\rho_n(x) = Const \times e^{-x^2} \langle|\det(xI_{n-1} - W_{n-1})|\rangle_{W_{n-1}}$$

2. Rank 1 deviations from unitarity ($0 < \gamma \leq 1$): $W_n = R_n U_n$, where

$$U_n \text{ is Haar unitary (CUE) and } R_n = \begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

Note that $W_n W_n^* = \begin{pmatrix} 1-\gamma & 0 \\ 0 & I_{n-1} \end{pmatrix}$.

Choice of $\gamma = 1$ corresponds to subunitary matrices (delete 1st row&column).

Mean density of eigenvalues of W_n:

$$\rho_n(x,y) = \begin{cases} \dfrac{n-1}{\pi\gamma|z|^2} \left(\dfrac{\tilde{\gamma}}{\gamma}\right)^{n-2} \langle|\det(zI_{n-1} - \tilde{R}_{n-1}U_{n-1})|^2\rangle_{U_{n-1}} & \text{if } 1-\gamma \leq |z|^2 \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\tilde{R}_{n-1} = \begin{pmatrix} \sqrt{1-\tilde{\gamma}} & 0 \\ 0 & I_{n-2} \end{pmatrix}, \qquad \tilde{\gamma} = \frac{|z|^2 + \gamma - 1}{|z|^2}.$$

[Also, rank-one deviations from Hermiticity - similar formula for $\rho_n(x,y)$]

Thm 1 *Let A be a complex matrix of size $n \times n$.*

(a) For any positive integer m:

$$\int_{U(n)} |\det(zI - AU)|^{2m} dU = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^m \det(|z|^2 I + t_j AA^*)\ d\mu_n(t_1, \ldots, t_m)$$

$$\text{where } d\mu_n = \frac{1}{c} \prod_{i<j} |t_i - t_j|^2 \prod_{j=1}^m \frac{dt_j}{(1+t_j)^{n+2m}}$$

(b) For any positive integer m such that $2m \le n$,

$$\int_{U(n)} \frac{dU}{|\det(zI - AU)|^{2m}} = \begin{cases} \int_0^1 \cdots \int_0^1 \dfrac{d\nu_n(t_1, \ldots, t_m)}{\prod_{j=1}^m \det(|z|^2 I - t_j AA^*)} & \text{if } |z|^2 > \lambda_{max}(AA^*), \\ \int_0^1 \cdots \int_0^1 \dfrac{d\nu_n(t_1, \ldots, t_m)}{\prod_{j=1}^m \det(I - t_j |z|^2 AA^*)} & \text{if } |z|^2 < \lambda_{min}(AA^*). \end{cases}$$

$$\text{where } d\nu_n = \frac{1}{\tilde{c}} \prod_{i<j} |t_i - t_j|^2 \prod_{j=1}^m (1 - t_j)^{n-2m} dt_j.$$

Thm 2 *For $n \geq 2$,*

$$\int_{U(n)} \frac{dU}{\det[\varepsilon^2 I_n + (I_n - zAU)(I_n - zAU)^*]}$$

$$= \frac{(n-1)}{2\pi i} \int_0^1 (1-t)^{n-2} dt \int_{-\infty}^{+\infty} \frac{dy}{y} \frac{1}{\det\left[|z|^2 AA^* + (\varepsilon^2 - t) I_n - i\varepsilon\sqrt{t}\left(y + \frac{1}{y}\right) I_n\right]}$$

If the eigenvalues a_j^2 of AA^* are distinct then, in the limit $\varepsilon \to 0$, the rhs is

$$-P_n(z) \log \varepsilon^2 + Q_n(z) + O(\varepsilon)$$

where

$$P_n(z) = (n-1) \sum_{j=1}^{n} (1 - |z|^2 a_j^2)^{n-2}\, \theta(1 - |z|^2 a_j^2) \prod_{k \neq j} \frac{1}{|z|^2 (a_k^2 - a_j^2)}$$

where θ is Heaviside's step fnc.

Note: For $\lambda_{min}(AA^*) \leq \frac{1}{|z|^2} \leq \lambda_{max}(AA^*)$ have log singularity ($P_n(z) \neq 0$).

If $\frac{1}{|z|^2} \leq \lambda_{min}(AA^*)$ or $\frac{1}{|z|^2} \geq \lambda_{max}(AA^*)$ then $P_n(z) = 0$.

Applications: Feinberg - Zee ring

Consider random A with inv. matrix distr. $dP(A, A^*) \propto e^{-n \operatorname{tr} V(AA^*)} dA dA^*$, $V(t)$ is a polynomial. If $V(t) = t$ have complex Gaussian matrices.

Note: A has complex eigvs, AA^* has real eigvs.

$$\left\langle |\det(zI_n - A)|^2 \right\rangle_A = \left\langle |\det(zI_n - AU)|^2 \right\rangle_A = (n+1) \int_0^{+\infty} \frac{\left\langle \det(|z|^2 + tAA^*) \right\rangle_A}{(1+t)^{n+2}} \, dt$$
$$= \exp[n\Phi(z) + o(n)]$$

where Φ is given in terms of limiting distribution, $d\sigma(\lambda)$, of eigvs of AA^*

$$\Phi(z) = \begin{cases} \log |z|^2 & \text{if } |z| > m_1 = \int \lambda d\sigma(\lambda), \\ \int_0^\infty \log \lambda d\sigma(\lambda) & \text{if } 1/|z| > m_{-1} = \int \frac{d\sigma(\lambda)}{\lambda}, \\ |z|^2 + \int_0^\infty \log \frac{\lambda + t_0}{|z|^2 + t_0} d\sigma(\lambda) & \text{if } 1/m_{-1} < |z| < m_1 \end{cases}$$

where t_0 is the (unique) solution of $\int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t} = \frac{1}{|z|^2 + t}$.

$$\lim_{n\to\infty} \left\langle \frac{1}{n} \log |\det(zI_n - AU)|^2 \right\rangle_A \overset{(?)}{=} \lim_{n\to\infty} \frac{1}{n} \log \left\langle |\det(zI_n - AU)|^2 \right\rangle_A = \Phi$$

$\Delta\Phi$ agrees with eigv density found by Feinberg & Zee.

Applications: Rank-one deviations from unitarity

Consider random $n \times n$ matrices W satisfying $WW^* = \begin{pmatrix} 1-\gamma & 0 \\ 0 & I_{n-1} \end{pmatrix}$.

Mean density of eigenvalues of W is given by:

$$\rho_n(|z|^2) = \frac{n-1}{\pi\gamma|z|^2} \left(\frac{\tilde{\gamma}}{\gamma}\right)^{n-2} \int_{U(n-1)} |\det(zI - \tilde{R}U)|^2 dU$$

where

$$\tilde{R} = \begin{pmatrix} \sqrt{1-\tilde{\gamma}} & 0 \\ 0 & I_{n-2} \end{pmatrix} \text{ and } \gamma = \frac{|z|^2 + \gamma - 1}{|z|^2}$$

Thm 1 comes in handy!

Interesting regime: $n \to \infty$ and $1 - |z|^2 \sim 1/n$

$$\{\text{expected no. of eigvs. of } A \text{ in } 1 - \tfrac{2\beta}{n} \le |z|^2 \le 1 - \tfrac{2\alpha}{n}\} = n \int_\alpha^\beta p(y)dy + o(n),$$

where $p(y) = -\left(\frac{\exp\{y(1-2/\gamma)\}}{y} \sinh y\right)'$

More general results:
Fyodorov-Sommers (finite-rank), Sommers-Zyczkowski (subunitary)

Sketch of proofs

$m = 1$:

$$\det(I_n + M) = \sum_{j=0}^{n} e_j(M) \text{ and } \int e_j(AU)\overline{e_k(AU)}dU = \frac{e_j(AA^*)}{e_j(I_n)}\delta_{j,k}$$

$$\Downarrow$$

$$\int \det(I_n + AU)(I_n + AU)^* dU = \sum_{j=0}^{n} \frac{e_j(AA^*)}{e_j(I_n)} = (n+1)\int_0^\infty \frac{\det(I_n + tAA^*)dt}{(1+t)^{n+2}}$$

$m = -1$: e_j replaced by h_j, complete sym. funcs.

Schur funcs. $s_\lambda = |h_{\lambda_i - i + j}|_1^n$, $s_{\lambda'} = |e_{\lambda_i - i + j}|_1^n$ – orthogonal on $U(n)$

$\det^m(I_n + M) = \sum_\lambda s_\lambda(I_m) s_{\lambda'}(M)$

$$\int \det^m(I_n + AU)(I_n + AU)^* dU = \sum_\lambda \frac{s_\lambda^2(I_m)}{s_{\lambda'}(I_n)} s_{\lambda'}(AA^*)$$

$\det^{-m}(I_n - M) = \sum_\lambda s_\lambda(I_m) s_\lambda(M)$

$$\int \det^{-m}(I_n - AU)(I_n - AU)^* dU = \sum_\lambda \frac{s_\lambda^2(I_m)}{s_\lambda(I_n)} s_\lambda(AA^*)$$

Useful identities (key element of proof of Thm 1):

Consider complex $m \times m$ matrices Z, volume element $dZdZ^* = \prod_{i,j} d\,\mathrm{Re}\, Z_{ij}\,\mathrm{Im}\, Z_{ij}$.

(a) if $l(\lambda) \le m$ and $l(\lambda') \le n$ then for any matrix $M$of size $m \times m$

$$\frac{1}{c_n} \int_{ZZ^* \geq 0} s_\lambda(MZZ^*) \frac{dZdZ^*}{\det(I_m + ZZ^*)^{n+2m}} = \frac{s_\lambda(M) s_\lambda(I_m)}{s_{\lambda'}(I_n)}$$

(b) if $l(\lambda) \le m$ and $2m \le n$ then for any matrix $M$of size $m \times m$

$$\frac{1}{\kappa_n} \int_{ZZ^* \le I_m} s_\lambda(MZZ^*) \det(I_m - ZZ^*)^{n-2m} dZdZ^* = \frac{s_\lambda(M) s_\lambda(I_m)}{s_\lambda(I_n)}$$

(b) can be derived from the Selberg integral:

$$\int_{[0,1]^m} s_\lambda(t_1, \ldots, t_m) \prod_{j=1}^{m} t_j^{a-1}(1-t_j)^{b-1} \prod_{i<j} (t_i - t_j)^2 dt_1 \ldots dt_m,$$

evaluations due to Kadell (1988), in more general settings (Jack polyn'ls) Kaneko (1993), Kadell (1997).

Not clear how to obtain (a) from this. We can prove (a), (b) independently.

Zirnbauer's color-flavor transformation (key element of proof of Thm 2)

Assume that $n \geq 2m$. Then for any two $n \times m$ matrices Φ and Ψ:

$$\int_{U(n)} e^{\mathrm{tr}\,\Phi^* U \Psi + \overline{\mathrm{tr}\,\Phi^* U \Psi}} dU = \int_{ZZ^* \leq I_m} e^{\mathrm{tr}\,\Phi Z \Phi^* + \overline{\mathrm{tr}\,\Psi Z \Psi^*}} \underbrace{d\nu_n(Z, Z^*)}_{Const \times \det(I_m - ZZ^*)^{n-2m} dZ dZ^*} \qquad (1)$$

The integral over Z can be evaluated with the help of identity (b), which yields

$$\int_{U(n)} e^{\mathrm{tr}\,\Phi^* U \Psi + \overline{\mathrm{tr}\,\Phi^* U \Psi}} dU = \frac{Const}{\prod_{i<j}(z_i^2 - z_j^2)} \times \det \left(\int_0^1 I_0\left(2\sqrt{t z_j^2}\right) t^{m-i} (1-t)^{n-2m} dt \right)_{i,j=1}^{m}$$

where z_j^2 are the eigenvalues of $\Psi^*\Psi\Phi^*\Phi$ and I_0 is the modified Bessel fnc.

Actually, Selberg $\Rightarrow$ Zirnbauer (bosonic). [(a) $\Rightarrow$ Zirnbauer (fermionic)?]

Use of Zirnbauer's formula [need its non-compact version provided by an integral representation of I_0] based on

$$\frac{1}{\det[\varepsilon^2 I_n + \Omega\Omega^*]} = \frac{1}{\det \begin{pmatrix} \varepsilon I_n & i\Omega \\ i\Omega^* & \varepsilon I_n \end{pmatrix}} = \frac{1}{\pi^n} \int_{\mathbb{C}^n} d^2\boldsymbol{p} \int_{\mathbb{C}^n} d^2\boldsymbol{q}\; e^{-[\varepsilon(|\boldsymbol{p}|^2 + |\boldsymbol{q}|^2) + i(\boldsymbol{q}^*\Omega^*\boldsymbol{p} + \boldsymbol{p}^*\Omega\boldsymbol{q})]}.$$

$\Omega = U^* - A$

Conclusions and outlook

- stochastic Horn problem (singular values $\rightsquigarrow$ eigenvalues)
- reproduce (but not prove) eigenvalue density. Have conjecture:

$$\frac{1}{n} < \log\det >= \frac{1}{n} \log < \det > \text{ (strong non-Hermiticity)}$$

- fractional moments or averages of ratios of spectral dets wanted

$$\text{mean eigv density.} = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \bar{z}} \lim_{z \to \zeta} \frac{\partial}{\partial \bar{\zeta}} \left\langle \frac{\det[\varepsilon^2 I + (zI - A)(zI - A)^*]}{\det[\varepsilon^2 I + (\zeta I - A)(\zeta I - A)^*]} \right\rangle$$

- other classical groups