LECTURE 6

Equation of Value; Worked Examples

Equation of Value

An equation of value equates the present value of cash inflows to the present value of cash outflows. If the present value of cash inflows is equal to the present value of cash outflows at a particular rate of interest, it means that the outgoing cash flows when invested with interest will generate the incoming cash flows.

The rate of interest at which the present value of outgoing cash flows is equal to the present value of incoming cash flows is the *yield* on the investment.

The equation of value can be expressed at any point of time. It brings together three quantities: amounts, time and a rate of interest. Given two of the quantities, the third is determined by the equation of value.

Worked Example 1.

A borrower is under an obligation to repay a bank £6280 in four years' time, £8460 in seven years' time and £7350 in thirteen years' time. As part of a review of his future commitments the borrower now offers either

(a) to discharge his liability for these three debts by making an appropriate single payment five years from now; or

(b) to repay the total amount owed (i.e. £22090) in a single payment at an appropriate time.

On the basis of an constant rate of interest 8% per annum effectively find the appropriate single payment if offer (a) is accepted by bank, and the appropriate time to repay the entire indebtedness if offer (b) is accepted.

Solution.

(a) Out: £C at $t=5$. In: £6280 at $t=4$, £8460 at $t=7$ and £7350 at $t=13$. Find C.

Equation of value (expressed at the present time, $t=0$): $Cv^5 = 6280v^4 + 8460v^7 + 7350v^{13}$.

Equation of value (expressed at $t=4$): $Cv = 6280 + 8460v^3 + 7350v^9$. ◄

$$\therefore \quad C = \frac{6280 + 8460v^3 + 7350v^9}{v}; \quad v = \frac{1}{1.08} = \frac{1}{1+i}\Big|_{@i=0.08}$$
$$= 18006.46 \text{ (a single payment of £18006)}$$

(b) Out: £22090 at time t. In: £6280 at $t=4$, £8460 at $t=7$ and £7350 at $t=13$. Find t.
Equation of value (expressed at the present time, $t=0$): $22090v^t = 6280v^4 + 8460v^7 + 7350v^{13}$.

$$\therefore \quad t = \frac{\ln \frac{6280v^4 + 8460v^7 + 7350v^{13}}{22090}}{\ln v}; \quad \left[v = \frac{1}{1.08} = \frac{1}{1+i}\Big|_{@i=0.08}\right]$$
$$= 7.66 \text{ (years)}$$

Worked Example 2.

Payments are made of £100 per annum at the beginning of each year for 10 years and an additional payment of £100 is made at an unknown time. The yield is 8% and the depositor receives £1700 after 10 years. What is the time (measured from outset) of the additional payment?

Solution.

Out: annuity-due of £100 p.a. and £100 at time t. In: £1700 at time $t=10$. Find t.
Equation of value at outset ($t=0$): $100\ddot{a}_{\overline{10|}} + 100v^t = 1700v^{10}$.

$$\therefore \quad t = \frac{\ln\left(17v^{10} - \ddot{a}_{\overline{10|}}\right)}{\ln v} \qquad \left[v = \frac{1}{1.08} = \frac{1}{1+i}\Big|_{@i=0.08}\right]$$
$$= 6.06 \text{ (years)} \qquad \left[\ddot{a}_{\overline{10|}} = 7.2469 = \frac{1-v^{10}}{1-v}\Big|_{@i=0.08}\right]$$

Worked Example 3.

A loan of £2400 is to be repaid by 20 equal annual installments. The rate of interest for the transaction is 10% per annum. Find the annual payment, assuming that payments are made (a) in advance and (b) in arrears.

Solution.

Bank. Out £: 2400 at time $t=0$. In: annuity of C per annum for 20 years (immediate annuity when payments made in arrear and annuity-due when payments made in advance). Find C.

(a) Equation of value at $t=0$: $C\,\ddot{a}_{\overline{20|}} = 2400$.

$$\therefore \quad C = \frac{2400}{\ddot{a}_{\overline{20|}}} \qquad \left[\ddot{a}_{\overline{20|}} = 9.3649 = \frac{1-v^{20}}{1-v}\Big|_{@i=0.10}\right]$$
$$= 256.28$$

(b) Equation of value at $t=0$: $Ca_{\overline{20|}} = 2400$.

$$\therefore \quad C = \frac{2400}{a_{\overline{20|}}} \qquad \left[a_{\overline{20|}} = v\,\ddot{a}_{\overline{20|}} = 8.5136, \quad v = \frac{1}{1.1} = \frac{1}{1+i}\Big|_{@i=0.10}\right]$$
$$= 281.90$$

Worked Example 4.

Given $\delta = 0.12$, find the values of d and $d^{(12)}$. What is the amount to be paid in advance for use of £1000 over one month?

Solution.

Use $d^{(p)} = p\left(1 - e^{-\frac{\delta}{p}}\right)$; $\quad d^{(12)} = 12(1 - e^{-\frac{0.12}{12}}) = 0.1194$, or $d^{(12)} = 11.94\%$

By the definition of $d^{(12)}$, we pay $\frac{d^{(12)}}{12}$ in advance for use of 1 over one month. Therefore, we should pay £$1000 \times \frac{d^{(12)}}{12} =$ £9.95 for use of £1000.

N.B. The treatment of problems involving nominal rates of interest is always considerably simplified by an appropriate choice of the time units. For example,

On the basis of a nominal rate of interest of 12% per annum convertible monthly, the present value of 1 due after t years is

$$\begin{aligned} v^t = \left(\frac{1}{1+i}\right)^t &= \left[\frac{1}{1+\frac{i^{(12)}}{12}}\right]^{12t} \\ &= \left[\frac{1}{1+\frac{0.12}{12}}\right]^{12t} = \left[\frac{1}{1+0.1}\right]^{12t} \\ &= v^{12t}\Big|_{\text{at } i=0.1} \end{aligned}$$

Thus if we adopt one month as our time unit and use $\frac{i^{(12)}}{12} = 0.01$ as the effective rate of interest per unit time, we correctly value growth of money.

To recapitulate: A nominal rate of interest $i^{(p)}$ convertible p times per year implies that the capital grows at the rate $\frac{i^{(p)}}{p}$ per one p-thly time interval (i.e. time interval of length $\frac{1}{p}$). The general rule to be used in conjunction with nominal rates is very simple. Choose as the time unit the period corresponding to the frequency with which the nominal rate is convertible and use $\frac{i^{(p)}}{p}$ as the effective rate of interest per unit time.

Worked Example 5.

Interest is convertible semi-annually. An investment of £100 provides a return of £215 in $15^1\!/_2$ years?. Find the nominal rate of interest.

Solution 1. Set (unit time) = 6 months. Then $15^1\!/_2$ years equal 31 unit time periods.

Let i be the effective rate of interest per unit time. Then we have the equation for i: $100(1+i)^{31} = 215$. The effective rate of interest per six months' time period is $\frac{i^{(2)}}{2}$, where $i^{(2)}$ is the nominal rate of interest convertible semi-annually. Therefore, $i^{(2)} = 2\left[\left(\frac{215}{100}\right)^{\frac{1}{31}} - 1\right] = 0.05$ (to 4 decimal places); $i^{(2)} = 5\%$

Solution 2. Thinking of one year as unit time period. In $15^1\!/_2$ years £100 will grow to £$100(1+i)^{15^1\!/_2}$, where i is the effective rate of interest per annum. Therefore

$$\begin{aligned} 215 &= 100(1+i)^{15^1\!/_2} \\ &= 100\left[1+\frac{i^{(2)}}{2}\right]^{31} \quad \text{by Eq. (9) in Lecture 2} \end{aligned}$$

This equation for $i^{(2)}$ is identical to that obtained in Solution 1.

Worked Example 6.

A debt of £5,000 is to be discharged over 2 years by equal installments, each of £P, made at 3 months' intervals, the first due now. Find P provided the interest is charged at rate 1% per quarter. Find the equivalent effective annual rate of interest.

Solution.

Set (unit time) = 3 months. The effective rate of interest per unit time is 1%.

Equation of value at $t = 0$: $\qquad 5000 = P\,\ddot{a}_{\overline{8}|@i=0.01}$.

$$\therefore \qquad P = 5000\left[\frac{1-v}{1-v^8}\Big|_{@i=0.01}\right] = 5000\left[\frac{1-\frac{1}{1.01}}{1-\left(\frac{1}{1.01}\right)^8}\right] = 646.98$$

The interest is compounded quarterly at nominal rate 4×1%=4% per annum, or $[(1+0.01)^4 - 1] \times$ 100%=4.06% per annum effectively.

Worked Example 7.

A vacuum cleaner can be purchased for £159.99 cash, or, equivalently by paying £20 down and then a certain amount P at the end of month over one year. Find the monthly payment if the rate of the interest charged is 24% nominal. Find the APR on this transaction.

Solution.

Set (unit time) = one month. Then the rate of interest per unit time is $\frac{i^{(12)}}{12} = \frac{0.24}{12} = 0.02$.

Equation of value at $t = 0$: $\qquad 159 = 20 + Pa_{\overline{12}|@i=0.02}$.

$$\therefore \qquad P = \frac{159-20}{a_{\overline{12}|@i=0.02}} = 139\left[\frac{1-v}{v(1-v^{12})}\Big|_{@i=0.02}\right] = 139\left[\frac{0.02}{1-\left(\frac{1}{1.02}\right)^{12}}\right] = 13.14$$

The interest is compounded monthly at nominal rate 24% per annum, or $[(1+0.02)^{12} - 1] \times$ 100%=26.82% per annum effectively. Therefore the APR is 26.82%.

Notice

$$\begin{aligned} \lim_{p\to\infty} a^{(p)}_{\overline{n}|} &= \lim_{p\to\infty} \frac{v^{\frac{1}{p}}(1-v^n)}{p(1-v^{\frac{1}{p}})} = (1-v^n)\lim_{x\to 0}\frac{xv^x}{1-v^x} \\ &= \frac{1-v^n}{-\ln v} \qquad \text{[by L'Hopital's rule]} \\ &= \overline{a}_{\overline{n}|} \qquad \text{[cf. Eq. (2) in Lecture 5]} \end{aligned}$$

Frequent payments ($p \to \infty$) in perpetuity ($n \to \infty$): $a^{(p)}_{\overline{\infty}|} \simeq \lim_{n\to\infty} \overline{a}_{\overline{n}|} = \frac{1}{-\ln v}$.

Worked Example 8.

Assuming no charges, what price should be paid for an annuity which provides £20 per week in perpetuity? Assume APR=10% and use continuous approximation of weekly payments.

Solution.

By the equation of value, the price should be equal to the present value of the annuity. The annuity provides £52×20 per annum, hence its present value is £1040 $a^{(52)}_{\overline{\infty}|}$. Using the approximation by continuous payment,

$$\text{Price} = £1040\, a^{(52)}_{\overline{\infty}|} \simeq £1040 \lim_{n\to\infty} \overline{a}_{\overline{n}|} = £1040 \frac{1}{-\ln v}\Big|_{@i=0.10} = £10912$$