LECTURE 4

Rates of Discount; Present Values

Assume $\delta(t)$, that is the force of interest changes over time.

Assume that interest is compounded p-thly. The interest for use of one unit of money over one sub-period of time of length $^1/_p$ starting at time t is due at the end of the sub-period, i.e. at time $t+{}^1/_p$. The amount due is $\frac{1}{p}i^{(p)}(t)$.

But equally we could pay for use of money at the start of the sub-period. Denote the amount of this equivalent payment by $\frac{1}{p}d^{(p)}(t)$, so that:

> $d^{(p)}(t)$ is the rate per *unit* time and per unit capital at which interest for use of money over time period $[t, t+{}^1/_p]$ is paid in advance.

$d^{(p)}(t)$ is called the *nominal rate of discount convertible p-thly*. When $p=1$ the notation d is used instead of $d^{(1)}$. Thus

> $d(t)$ is the rate at which interest for use of money over unit time period is paid in advance. When unit time is one year d is called the annual rate of discount.

Nominal rates of discount can be converted to an *effective annual rate of discount*; see Eq. (2).

When we borrow one unit of money for use over one sub-period and pay interest in advance, effectively we receive $1-\frac{1}{p}d^{(p)}(t)$ at time t and repay 1 at time $t+{}^1/_p$. Thus, for example,

$$1-d(0)=\left[1-\frac{d^{(4)}(\frac{3}{4})}{4}\right]\left[1-\frac{d^{(2)}(\frac{1}{4})}{2}\right]\left[1-\frac{d^{(4)}(0)}{4}\right].$$

This can be seen by working backwards through the year.

We have defined $A(t_1,t_2)$ as the accumulation at time t_2 of one unit of money invested at time t_1. Now define $D(t_1,t_2)$ as the value of an investment at time t_1 which give a return of one unit of money at time t_2, i.e.

$$D(t_1,t_2)=\frac{1}{A(t_1,t_2)}.$$

The nominal rates of interest and discount are related: payment of $\frac{1}{p}d^{(p)}(t)$ at time t is equivalent to payment of $\frac{1}{p}i^{(p)}(t)$ at time $t+{}^1/_p$. Or, in other words, an investment of $\frac{1}{p}d^{(p)}(t)$ at time t gives a return of $\frac{1}{p}i^{(p)}(t)$ at time $t+{}^1/_p$. Therefore

$$\frac{d^{(p)}(t)}{p}=\frac{i^{(p)}(t)}{p}D\Big(t,t+\frac{1}{p}\Big)=\frac{i^{(p)}(t)}{p}\frac{1}{1+\frac{i^{(p)}(t)}{p}}, \qquad \text{[by Eq. (4) in Lecture 2]} \qquad (1)$$

and (explain why)

$$\delta(t)=\lim_{p\to\infty}d^{(p)}(t).$$

Back to constant force of interest, $\delta(t)=\delta$ for all t.

In this case $D(t_0,t_0+t)=e^{-t\delta}$ independently of t_0. In particular,

$$1-\frac{d^{(p)}}{p}=D\left(0,\frac{1}{p}\right)=e^{-\frac{\delta}{p}},$$

hence

$$d^{(p)}=p(1-e^{-\frac{\delta}{p}}). \qquad \text{(therefore } d^{(p)} \text{ does not change over time)}$$

By the principle of consistency (working backwards in time)

$$\begin{aligned} D(0,1) &= D\left(\frac{p-1}{p},1\right)D\left(\frac{p-2}{p},\frac{p-1}{p}\right)\cdots D\left(0,\frac{1}{p}\right) \\ &= \left[1-\frac{d^{(p)}}{p}\right]^p. \qquad \left[\text{as } D\left(\frac{k}{p},\frac{k+1}{p}\right)=1-\frac{d^{(p)}}{p} \text{ for any } k\right] \end{aligned}$$

Notice that $D(0,1)=1-d$. Therefore

$$1-d=\left[1-\frac{d^{(p)}}{p}\right]^p. \qquad (2)$$

This equation (compare it to Eq. (9) in Lecture 2) relates the nominal rate of discount convertible p-thly to the equivalent effective annual rate of discount.

Standard notation:

$$v=\frac{1}{1+i}, \text{ where } i \text{ is the effective annual rate}$$

Notice that (can you interpret these relations in words?)

$$d(1+i)=i; \quad d=\frac{i}{1+i}=vi.$$

If you invest £Cv^t at time $t=0$ (the present time) then you will get a return of £C at time t (as $(Cv^t)\times(1+i)^t=C$).

Cv^t is called the (discounted) *present value* of C due at time t.

Notice that $1-d=v$; this is to say that $1-d$ is the discounted present value, at time $t=0$, of 1 due at time t.

Two rules to remember:

Suppose the value of investment at the present time is C. Then:

- to find the value of investment after t years you multiply C by $(1+i)^t$;
- to find the value of investment t years before the present time you multiply C by v^t .

Two sorts of cash payments (cash flows):- discrete and continuous.

Discrete cash flows.

The present value (i.e. at time $t=0$) of payments of $c_1, c_2, \ldots, c_n$ due at times $t_1, t_2, \ldots, t_n$ in the future is

$$c_1 v^{t_1} + c_2 v^{t_2} + \ldots + c_n v^{t_n} = \sum_{j=1}^{n} c_j v^{t_j} \tag{3}$$

Continuously payable cash flows (payment streams).

The present value (i.e. at time $t=0$) of a continuous payment at rate $\rho(t)$ over a time interval $[t_1, t_2]$ in the future is

$$\int_{t_1}^{t_2} v^t \rho(t) dt \qquad \left(\text{the same as } \int_{t_1}^{t_2} e^{-t\delta} \rho(t) dt\right), \tag{4}$$

where $\rho(t)$ is the rate (per unit time) at which the payment is made at time t, i.e. $\rho(t) = \lim_{h \downarrow 0} \frac{1}{h} \times$(amount paid over $[t, t+h]$).

Continuous payment at rate $\rho(t) = \rho$ over time period [0,1]: present value $= iv\frac{\rho}{\delta}$ (derive this!)

We say that two payments are *equivalent* if their present values coincide.

Five equivalent ways of paying interest on a loan of 1 unit of money over $[0,1]$:

	t=0	t=$\frac{1}{p}$	t=$\frac{2}{p}$	...	t=$\frac{p-1}{p}$	t=1	Time, t
(1)	d						
(2)	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\ldots$	$\frac{d^{(p)}}{p}$		
(3)		$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	$\ldots$	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	Payments
(4)						i	
(5)	$\longleftarrow$	Continuous payment at rate δ				$\longrightarrow$	

Notice that

- the present value of continuous payment at constant rate $\rho(t) = \delta$ over [0,1] (entry (5) in the table above) is $iv\frac{\rho}{\delta}\Big|_{\rho=\delta} = iv$,
- the present value of a single payment of i at time $t=1$ (entry (4) in the table) is iv,
- the present value of a single payment of d at time $t=0$ (entry (1) in the table) is d and $d = iv$.

Therefore (1) $\Leftrightarrow$(5)$\Leftrightarrow$(4).

Exercise: obtain (3)$\Leftrightarrow$(4)$\Leftrightarrow$(2).

SUMMARY.	
Discounted value at time t_1 of a unit invst. at time t_1:	$D(t_1, t_2) = \exp\left[-\int_{t_1}^{t_2} \delta(s) ds\right]$
When interest is converted p-thly:	$D\left(t, t+\frac{1}{p}\right) = 1 - \frac{d^{(p)}(t)}{p}$
Constant force of interest δ:	$1-d = \left[1 - \frac{d^{(p)}}{p}\right]^p$, $d = iv$
Discounted present value of C due in time t:	cv^t, where $v = \frac{1}{1+i} = 1-d$