LECTURE 17

Why does one need to know var(P.V.)?

Suppose that we have a block of identical whole life assurances, each with a sum assured of S units of money, sold by a life office to N independent lives age x whose mortality is described by a common survival function.

Let us label the lives by j, $j=1,\dots N$, so that T_j is the time-until-death for life j. Then $z_j = Sv^{T_j}$ is the present value of the death benefit payable immediately on the death of life j. The total present value of the death benefits is $Z = z_1 + z_2 + \ldots + z_N$ and

$$E(Z) = \sum_{j=1}^{N} E(z_j) = NS\bar{A}_x.$$

The lives are independent. Therefore,

$$\mathrm{var}(Z) = \sum_{j=1}^{N} \mathrm{var}(z_j) = NS^2\left[\bar{A}_x^* - (\bar{A}_x)^2\right].$$

Notice that the expected value of Z is of the order of N and the standard deviation, $\sqrt{\mathrm{var}(Z)}$, is of the order of $\sqrt{N}$, i.e. relatively small.

Accuracy of estimating of Z *by* $E(Z)$. (non-examinable)

By the Chebyshov inequality, $P(|Z-E(Z)|>\varepsilon) \le \frac{\mathrm{var}(Z)}{\varepsilon^2}$. Hence $P(|Z-E(Z)|\le\varepsilon) \ge 1-\frac{\mathrm{var}(Z)}{\varepsilon^2}$ Set

$$\varepsilon = \sqrt{\frac{\mathrm{var}(Z)}{0.05}} = S\sqrt{N}\sqrt{\frac{\bar{A}_x^* - (\bar{A}_x)^2}{0.05}}.$$

Then we are 95% confident that the following inequality holds:

$$\bar{A}_x - \frac{1}{\sqrt{N}}\sqrt{\frac{\bar{A}_x^* - (\bar{A}_x)^2}{0.05}} \le \frac{Z}{SN} \le \bar{A}_x + \frac{1}{\sqrt{N}}\sqrt{\frac{\bar{A}_x^* - (\bar{A}_x)^2}{0.05}},$$

that is the total present value of the benefits under N identical policies per unit sum assured and per individual fluctuates about $\bar{A}_x$ with the magnitude of fluctuations being proportional to $\frac{1}{\sqrt{N}}$.

Normal Approximation: (examinable)

when N is large $\dfrac{Z-E(Z)}{\sqrt{\mathrm{var}(Z)}}$ can be approximated by $N(0,1)$, so that

$$P\left(\frac{Z-E(Z)}{\sqrt{\mathrm{var}(Z)}} \le u\right) \simeq P(N(0,1)\le u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-\frac{t^2}{2}}\,dt = \Phi(u)$$

Worked Example 1. $\bar{A}_x$ under the assumption of constant force of mortality

Assume a constant force of mortality μ and a constant force of interest δ. Express the expectation and variance of the present value of unit death benefit, payable immediately on death under a whole life assurance policy, in terms of μ and δ.

Solution.

$\mu(u) = \mu$ for all $u \geq 0$. Therefore $s(x) = e^{-\int_0^x \mu du} = e^{-\mu x}$ for all $x \geq 0$, and ${}_tp_x = \frac{s(x+t)}{s(t)} = e^{-\mu t}$ for all $t \geq 0$ (notice that ${}_tp_x$ does not depend on x; this is a property specific to constant force of mortality).
The present value of the benefit is $v^{T(x)}$ and

$$\begin{aligned} \bar{A}_x = E(v^{T(x)}) &= \int_0^\infty v^t\, {}_tp_x \mu(x+t)\, dt \\ &= \int_0^\infty e^{-\delta t} e^{-\mu t} \mu\, dt \qquad \left[\text{as} \quad v = \frac{1}{1+i} = e^{-\delta}\right] \\ &= \frac{\mu}{\mu+\delta}. \end{aligned} \tag{1}$$

The variance of the P.V. v^T is $\bar{A}_x^* - (\bar{A}_x)^2$, where $v^* = v^2 = e^{-2\delta}$. Hence $\delta^* = 2\delta$ and

$$\operatorname{var}(v^{T(x)}) = \frac{\mu}{\mu+2\delta} - \left(\frac{\mu}{\mu+\delta}\right)^2 \tag{2}$$

Worked Example 2.

Suppose that a life office is about to sell the whole life assurance policy with unit death benefit to 100 individuals, age x, who were born on the same day.
The death benefits are to be paid on the moment of death. They are to be withdrawn from an investment fund earning interest i, such that $\delta = \ln(1+i) = 0.06$.
Assume that the lives assured are subject to a constant force of mortality $\mu = 0.04$ and calculate the minimum amount to be invested at time $t = 0$, so that the probability is approximately 0.95 that sufficient funds will be on hand to withdraw the benefit payment at the death of each individual.
Assume that the future lifetimes for the lives assured are independent random variables.

Solution

The minimum amount is such a value of h that $P(Z \leq h) = 0.95$, where Z is the total present value of death benefits, $Z = \sum_{j=1}^{100} z_j$ and $z_j = v^{T_j}$.

$$\begin{aligned} E(Z) &= N\bar{A}_x \qquad [\text{by using (1)}] \\ &= N\frac{\mu}{\mu+\delta} = 100 \times \frac{0.04}{0.04+0.06} = 40. \end{aligned}$$

$$\begin{aligned} \operatorname{var}(Z) &= N\left[\bar{A}_x^* - (\bar{A}_x)^2\right] \qquad [\text{using independence of } T_j \text{ and then (2)}] \\ &= N\left[\frac{\mu}{\mu+2\delta} - \left(\frac{\mu}{\mu+\delta}\right)^2\right] = 100 \times \left[\frac{0.04}{0.04+2\times 0.06} - \left(\frac{0.04}{0.04+0.06}\right)^2\right] = 9. \end{aligned}$$

We want to find such value of h that $P(Z \leq h) = 0.95$. Obviously,

$$P(Z \leq h) = P\left(\frac{Z - E(Z)}{\sqrt{\operatorname{var}(Z)}} \leq \frac{h - E(Z)}{\sqrt{\operatorname{var}(Z)}}\right).$$

Therefore, by using the normal approximation we obtain an equation for h:

$$\begin{aligned} 0.95 &= P\left(\frac{Z - E(Z)}{\sqrt{\operatorname{var}(Z)}} \leq \frac{h - E(Z)}{\sqrt{\operatorname{var}(Z)}}\right) \\ &\simeq P\left(N(0,1) \leq \frac{h - E(Z)}{\sqrt{\operatorname{var}(Z)}}\right) = \Phi\left(\frac{h - E(Z)}{\sqrt{\operatorname{var}(Z)}}\right). \end{aligned}$$

From statistical tables we find that $\Phi(u) = 0.95$ when $u = 1.645$, therefore

$$\frac{h - E(Z)}{\sqrt{\operatorname{var}(Z)}} \simeq 1.645$$

or

$$\begin{aligned} h &\simeq E(Z) + 1.645 \times \sqrt{\operatorname{var}(Z)} \\ &= 40 + \sqrt{9} \times 1.645 = 44.935. \end{aligned}$$

Therefore one has to invest at least £44.94 to be 95% confident that sufficient funds will be on hand to pay each of the benefit payments.
If the sum assured is £10,000 then by proportion the minimum amount required is £10,000$\times$ 44.935=£44,935.