LECTURES 14-15

Interpolation of life-table values for fractional ages

Linear Interpolation:

(has been used in the derivation of $\mathring{e}_x = e_x + 1/2$)

If we know value of $s(x)$ only for integer x we can approximate $s(y)$ in the interval between x and $x+1$ by the linear function:

$$s(x+t) \simeq (1-t)s(x) + ts(x+1), \quad x = 0,1,2,\ldots, 0 < t < 1.$$

This approximation implies that

$$\begin{aligned}
{}_{k+t}p_x &\simeq {}_k p_x\,(1-t) + {}_{k+1}p_x\, t \qquad x,k = 0,1,2,\ldots, \quad 0 < t < 1. \\
l_{x+t} &\simeq (1-t)l_x + t l_{x+1}, \qquad x = 0,1,2,\ldots, \quad 0 < t < 1, \\
&= l_x - t d_x.
\end{aligned}$$

The expected number of deaths in the interval $[x+t, x+t+\Delta t]$ is

$${}_{\Delta t}d_{x+t} = l_{x+t} - l_{x+t+\Delta t}.$$

Applying linear interpolation when Δt is small,

$${}_{\Delta t}d_{x+t} \simeq l_x - td_x - [l_x - (t+\Delta t)d_x] = \Delta t d_x.$$

This means that the density of deaths within the age interval between x and $x+1$ is d_x, i.e. constant. Therefore linear interpolation implies uniform distribution of deaths within each year of age. The inverse statement is also true: uniform distribution of deaths within each year of age implies piece-wise linear survival function.

Therefore, linear interpolation is consistent with the assumption of uniform distribution of deaths within each year of age.

Also in use in actuarial science:-

Exponential interpolation

$$\ln s(x+t) \simeq (1-t)\ln s(x) + t\ln s(x+1), \quad x = 0,1,2,\ldots \text{ and } 0 < t < 1.$$

Exponential interpolation is consistent with the assumption of a constant force of mortality within each year of age.

Harmonic interpolation (also known as hyperbolic or Balducci assumption)

$$\frac{1}{s(x+t)} \simeq (1-t)\frac{1}{s(x)} + t\frac{1}{s(x+1)}, \quad x = 0,1,2,\ldots \text{ and } 0 < t < 1.$$

Force of mortality cannot be observed. Its values (even those for integer x given in life tables) has to be calculated from the values of other life-table functions. Linear interpolation is not suitable for this purpose as it gives two different values of $\mu(x)$, one when it is used over the interval $[x-1,x]$ and the other when it is used over $[x,x+1]$. Because of this other approximations are usually used. Four most common are:

(1) Based on $p_x = e^{-\int_0^1 \mu(x+t)dt}$.

$$\int_0^1 \mu(x+1) = -\ln p_x \qquad \therefore \quad \mu\big(x+\frac{1}{2}\big) \simeq -\ln p_x$$

(2) Based on ${}_2p_{x-1} = e^{-\int_{-1}^1 \mu(x+t)dt}$ (follows from ${}_2p_{x-1} = p_{x-1}p_x$).

$$-\ln(p_{x-1}p_x) = \int_{-1}^1 \mu(x+t)dt \simeq 2\mu(x) \qquad \therefore \quad \mu(x) \simeq -\frac{1}{2}\Big(\ln p_x + \ln p_{x-1}\Big)$$

(3) Based on the assumption that l_y is a quadratic polynomial in y in the interval $[x-1,x+1]$.

$$\mu(x) \simeq \frac{l_{x-1}-l_{x+1}}{l_x}.$$

(4) Based on the assumption that l_y is a quartic polynomial in y in the interval $[x-2,x+2]$.

$$\mu(x) \simeq \frac{8(l_{x-1}-l_{x+1})-(l_{x-2}-l_{x+2})}{12l_x}.$$

Example.

Using approximation (4) and the values of l_x for ages 38,39,40,41,42 from the English Life Table No. 12 – Males, estimate $\mu(40)$

$$\begin{aligned}\mu(x) &\simeq \frac{8(l_{x-1}-l_{x+1})-(l_{x-2}-l_{x+2})}{12l_x} \\ &= \frac{8(93991-93570)-(94176-93328)}{12\times 93790} = 0.00224.\end{aligned}$$

The obtained value coincides with the value of μ_{40} in the table.