MAS224, Actuarial Mathematics: Solutions to Problem Sheet 9.

1. The Leslie matrix has $b_0,\ b_1$ and b_2 in the first row, i.e. $1,\ 4$ and 0. It has p_0 and p_1 (i.e. $\frac{1}{2}$ and $\frac{1}{3}$) just below the main diagonal. So

$$\mathbf{M} = \begin{pmatrix} 1 & 4 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$$

$S(0) = 1$, $S(1) = p_0 = \frac{1}{2}$ and $S(2) = p_0 \times p_1 = \frac{1}{6}$.

The expected number of female offspring produced during the lifetime of a single female beetle is just $\sum_{x=0}^{2} b_x S(x) = S(0) + 4S(1) = 3$. Since this is not equal to 1, there is no solution with $n_x(t) \equiv n_x$.

Either: The Euler solution has $n_x(t) = \lambda^t n_x$, with $n_x = CS(x)/\lambda^x$ for any positive C, where λ is the unique positive solution to $\sum_{x=0}^{2} \frac{b_x S(x)}{\lambda^{x+1}} = 1$.

The last condition is just $\lambda^2 = \lambda + 2$ which has roots -1 and 2. Hence $\lambda = 2$. Therefore the Euler solution has $n_x(t) = 2^t n_x$ where $n_0 = C$, $n_1 = C/4$ and $n_2 = C/24$ for any positive C.

Or: The Euler solution has $\mathbf{n}(t) = \lambda^t \mathbf{n}$ where λ is the unique positive eigenvalue of $\mathbf{M}$ and $\mathbf{n}$ is a right eigenvector of $\mathbf{M}$ corresponding to λ. (Note that the eigenvector is unique up to a multiple and the multiple must make the vector non-negative and non-zero.)

Now

$$|\mathbf{M} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 4 & 0 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{3} & -\lambda \end{vmatrix} = (1-\lambda)\lambda^2 - \tfrac{1}{2}(-4\lambda) = -\lambda(\lambda+1)(\lambda-2)$$

Hence the eigenvalues are $-1, 0, 2$. So the Euler solution has $\lambda = 2$. The right eigenvector has to satisfy $-n_0 + 4n_1 = 0$, $(1/2)n_0 - 2n_1 = 0$ and $(1/3)n_1 - 2n_2 = 0$. Hence $n_0 = 24B$, $n_1 = 6B$ and $n_2 = B$ for any positive B. This is the same solution obtained using the previous method if we let $B = C/24$.

Last part

If $\mathbf{n}(0) = \begin{pmatrix} 900 \\ 900 \\ 900 \end{pmatrix}$ then $\mathbf{n}(1) = \mathbf{M}\mathbf{n}(0) = \begin{pmatrix} 4500 \\ 450 \\ 300 \end{pmatrix}$ and so $\mathbf{n}(2) = \mathbf{M}\mathbf{n}(1) = \begin{pmatrix} 6300 \\ 2250 \\ 150 \end{pmatrix}$

and then $\mathbf{n}(3) = \mathbf{M}\mathbf{n}(2) = \begin{pmatrix} 15300 \\ 3150 \\ 750 \end{pmatrix}$

(This is an example where doing a few iterations does not enable you to see the long term behaviour.)

2. The Leslie matrix has $b_0,\ b_1$ and b_2 in the first row, i.e. $0,\ 2$ and 0. It has p_0 and p_1 (i.e. $\frac{1}{2}$ and $\frac{1}{4}$) just below the main diagonal. So

$$M = \begin{pmatrix} 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

(i) $S(0) = 1$, $S(1) = p_0 = 1/2$ and $S(2) = p_0 p_1 = 1/8$. Also $b_0 = b_2 = 0$ and $b_1 = 2$. Hence $\sum_{x=0}^{2} b_x S(x) = 1$.

$0 = |\mathbf{M} - \lambda\mathbf{I}| = -\lambda^3 + 2\lambda = -\lambda(\lambda-1)(\lambda+1)$. Therefore the eigenvalues are $0, -1$ and 1 and so 1 is the only real positive eigenvalue of $\mathbf{M}$.

The Euler solution can be found either by finding the right eigenvector directly (it is unique up to a multiple), or by using the result that the eigenvector is a multiple of

$$\begin{pmatrix} S(0) \\ S(1) \\ S(2) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{8} \end{pmatrix}$$

Hence the Euler solution has $n_0(t) = C$, $n_1(t) = C/2$ and $n_2(t) = C/8$ for all t and any positive constant C.

(ii) $\mathbf{M}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{8} & 0 & 0 \end{pmatrix}$ and so $\mathbf{M}^3 = \begin{pmatrix} 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$ Therefore $\mathbf{M}^3 = \mathbf{M}$ as required.

If $\mathbf{n}(0) = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$ then $\mathbf{n}(1) = \mathbf{M}\mathbf{n}(0) = \begin{pmatrix} 200 \\ 50 \\ 25 \end{pmatrix}$ and so $\mathbf{n}(2) = \mathbf{M}\mathbf{n}(1) = \begin{pmatrix} 100 \\ 100 \\ 12.5 \end{pmatrix}$

Since $\mathbf{M}^3 = \mathbf{M}$, $\mathbf{n}(3) = \mathbf{M}^3\mathbf{n}(0) = \mathbf{M}\mathbf{n}(0) = \mathbf{n}(1)$ and so $\mathbf{n}(4) = \mathbf{M}\mathbf{n}(3) = \mathbf{M}\mathbf{n}(1) = \mathbf{n}(2)$. Hence it is easily seen that the population structure oscillates between two values for $t \geq 1$ with $\mathbf{n}(2t+1) = \mathbf{n}(1)$ and $\mathbf{n}(2t+2) = \mathbf{n}(2)$ for all $t \geq 1$.

This gives an example when $\mathbf{n}(t)$ does not tend to the Euler solution.

3. The equation is $\sum_{x=0}^{\infty} \frac{b_x S(x)}{\lambda^{x+1}} = 1$ where $S(x) = p_0 \times p_1 \times \ldots \times p_{x-1}$.

(i) $S(x) = e^{-\theta x}$ and $b_x = e^{-\alpha x}$. Hence the equation for λ is just

$$1 = \sum_{x=0}^{\infty} \frac{e^{-\alpha x} e^{-\theta x}}{\lambda^{x+1}} = \frac{\frac{1}{\lambda}}{\left(1 - \frac{e^{-(\alpha+\theta)}}{\lambda}\right)} = \frac{1}{\lambda - e^{-(\alpha+\theta)}}$$

and so $\lambda = 1 + e^{-(\alpha+\theta)}$.

Since $\lambda > 1$, the population is growing for any values of α and θ.

(ii) $S(x) = e^{-\theta x}$ and $b_0 = 0$ and $b_x = \alpha$ for $x = 1, 2, \ldots$. Hence the equation for λ is just

$$1 = \sum_{x=1}^{\infty} \frac{\alpha e^{-\theta x}}{\lambda^{x+1}} = \frac{\frac{\alpha e^{-\theta}}{\lambda^2}}{\left(1 - \frac{e^{-\theta}}{\lambda}\right)} = \frac{\alpha}{\lambda^2 e^{\theta} - \lambda}$$

Hence $e^{\theta}\lambda^2 - \lambda - \alpha = 0$. Since the root we require is positive, $\lambda = \frac{1+\sqrt{1+4\alpha e^{\theta}}}{2e^{\theta}}$.

The population will be growing if $\lambda > 1$. We can find a condition for this to occur in two ways:

<u>Either:</u> From the equation for λ, as in lectures, the positive solution will be greater than 1 if $\sum_{x=0}^{\infty} b_x S(x) > 1$, i.e. if $\sum_{x=1}^{\infty} \alpha e^{-\theta x} > 1$, i.e. if $\frac{\alpha e^{-\theta}}{1-e^{-\theta}} > 1$. This occurs precisely when $\alpha > e^{\theta} - 1$.

<u>Or:</u> Find the condition using the value of λ calculated earlier. So the condition for $\lambda > 1$ is that $1 + \sqrt{1+4\alpha e^{\theta}} > 2e^{\theta}$, i.e. $1 + 4\alpha e^{\theta} > (2e^{\theta} - 1)^2 = 4e^{2\theta} - 4e^{\theta} + 1$, i.e. $\alpha > e^{\theta} - 1$.

So the population is growing if $\alpha > e^{\theta} - 1$.