

MAS224, Actuarial Mathematics: Solutions to Problem Sheet 8.

- 1*. Let $\mathcal{L}P$ be the monthly payment. Then P is determined by the equation of (expected) value $100000 = 12P\ddot{a}_{[65]}^{(12)}$.

By making use of the approximation $\ddot{a}_x^{(p)} \approx \ddot{a}_x - \frac{p-1}{2p} = \frac{N_x}{D_x} - \frac{p-1}{2p}$,

$$P = \frac{100000}{12 \times \ddot{a}_{[65]}^{(12)}} = \frac{100000}{12 \left(\frac{N_{[65]}}{D_{[65]}} - \frac{11}{24} \right)} = \frac{100000}{12 \left(\frac{22940.948}{2087.5676} - \frac{11}{24} \right)} = 791.32$$

Thus £100,000 buys £791.32 per month paid in advance for life for a 65 old.

- 2*. Compare the immediate prize of £10,000 with the expected present value of the monthly payments, which constitute a life annuity-due of £100 paid monthly (so £1200 per annum). The expected present value in pounds is just:

$$1200\ddot{a}_{60}^{(12)} \approx 1200 \left(\ddot{a}_{60} - \frac{11}{24} \right) = 1200 \left(\frac{N_{60}}{D_{60}} - \frac{11}{24} \right) = 1200 \left(\frac{35841.261}{2855.5942} - \frac{11}{24} \right) = 14511.49$$

Since the expected present value of £100 per month is £14,511.49, which is greater than £10,000 it is preferable to take £100 per month.

- 3*. (a) If the annual premium is $\mathcal{L}P$ then $P\ddot{a}_{[50]} = 20,000A_{[50]}$, where $\ddot{a}_{[50]} = \frac{N_{[50]}}{D_{[50]}}$. By making use of the conversion relationship $A_x = 1 - d\ddot{a}_x$,

$$P = \frac{20,000 \times A_{[50]}}{\ddot{a}_{[50]}} = 20,000 \left(\frac{D_{[50]}}{N_{[50]}} - d \right) = 20,000 \left(\frac{4581.3224}{73544.823} - \frac{4}{104} \right) = 476.63$$

The annual premium is £476.63.

(b) On what would have been his 61st birthday, the actual loss is just (the value at that time of the benefit actually paid) - (the value at that time of the payments he has actually made), i.e. $20,000 - (1+i)^{11} \times P\ddot{a}_{\overline{11}|}$ pounds, where $P = 476.63$ was the premium found in part (a). So the loss in pounds is

$$\begin{aligned} 20,000 - (1+i)^{11} \times P \frac{1-v^{11}}{1-v} &= 20,000 - \frac{P(1+i)((1+i)^{11} - 1)}{i} \\ &= 20,000 - 476.63 \frac{(1.04)((1.04)^{11} - 1)}{0.04} \\ &= 13,314.88 \end{aligned}$$

(c) Ignoring expenses, the surrender value is the same as the policy value, and the latter is (the expected value, on John's 60th birthday, of the future benefit payment) - (the expected value, on John's 60th birthday, of the future premiums).

The expected value, on John's 60th birthday, of the future benefit payment (in pounds) is $20,000A_{[50]+10} = 20,000A_{60} = 20,000(1 - d\ddot{a}_{60})$.

The expected value, on John's 60th birthday, of the future premiums is (in pounds) $P\ddot{a}_{60}$, where P is the annual premium found in part (a).

We have $\ddot{a}_{60} = \frac{N_{60}}{D_{60}} = \frac{35841.261}{2855.5942}$.

Thus the surrender value of John's policy is

$$\begin{aligned} 20000(1 - d\ddot{a}_{60}) - P\ddot{a}_{60} &= 20000 \left(1 - \frac{4}{104} \frac{35841.261}{2855.5942} \right) - 476.63 \frac{35841.261}{2855.5942} \\ &= 4362.87 \quad (\text{to 2 d.p.}) \end{aligned}$$

Alternatively, the surrender value of John's policy is $\pounds 20,000 \times ({}_{10}V_{[50]})$, where ${}_mV_x = 1 - \ddot{a}_{x+m}/\ddot{a}_x$, a result derived in lectures. This gives the following answer

$$200000 \left(1 - \frac{\ddot{a}_{60}}{\ddot{a}_{[50]}} \right) = 200000 \left(1 - \frac{35841.261 \times 4581.3224}{2855.5942 \times 73544.823} \right) = 4362.92 \quad (\text{to 2 d.p.})$$

The discrepancy between two answers is due to the rounding off error in part (a).

- 4*. (a) Use the indicator-of-survival random variables I_j and follow the derivation of $\ddot{a}_{60}^{(p)}$ given in Lecture 25.

The P.V. of the annuity in question is $Z = \frac{1}{12} \sum_{j=0}^{12n-1} v^{j/12} I_j$, where I_j takes value 1 if (x) survives to age $x + \frac{j}{12}$ and 0 otherwise. The random variable I_j has Bernoulli distribution with parameter $\frac{j}{12} p_x$. Thus

$$\begin{aligned} \ddot{a}_{x:n}^{(12)} &= E(Z) = \frac{1}{12} \sum_{j=0}^{12n-1} v^{j/12} E(I_j) = \frac{1}{12} \sum_{j=0}^{12n-1} v^{j/12} \frac{j}{12} p_x \quad [\text{as } E(I_j) = \frac{j}{12} p_x] \\ &= \frac{1}{12} \sum_{j=0}^{12n-1} \frac{v^{x+\frac{j}{12}}}{v^x} \times \frac{l_{x+\frac{j}{12}}}{l_x} \\ &= \frac{1}{12} \sum_{k=0}^{12n-1} \frac{D_{x+\frac{j}{12}}}{D_x} \quad [\text{as } D_{x+t} = v^{x+t} l_{x+t}]. \end{aligned}$$

- (b) Follow the derivation of $A_x^{(p)}$ given in Lecture 25.

Let Z_1 and Z_2 be the present values of, correspondingly, the death and survival benefits in question. Then $A_{x:n}^{(12)} = E(Z_1) + E(Z_2)$.

Z_1 is the present value of the death benefit in question. Z_1 takes values $v^{\frac{j+1}{12}}$, $j = 0, 1, \dots, 12n-1$, with probabilities $P(\frac{j}{12} < T(x) < \frac{j+1}{12}) = \frac{j}{12} p_x - \frac{j+1}{12} p_x$, and it takes value 0 with probability $P(T(x) > n)$. It follows from this that

$$E(Z_1) = \sum_{j=0}^{12n-1} v^{\frac{j+1}{12}} P\left(\frac{j}{12} < T(x) < \frac{j+1}{12}\right) = \sum_{j=0}^{12n-1} v^{\frac{j+1}{12}} \left(\frac{j}{12} p_x - \frac{j+1}{12} p_x \right).$$

Z_2 takes value v^n if $P(T(x) > n)$ and 0 otherwise, so that it takes values v^n and 0 with probabilities ${}_n p_x$ and $1 - {}_n p_x$, correspondingly. It follows from this that

$$E(Z_2) = v^n \times ({}_n p_x).$$

Therefore

$$\begin{aligned} A_{x:n}^{(12)} &= \sum_{j=0}^{12n-1} v^{\frac{j+1}{12}} \frac{j}{12} p_x - \sum_{j=0}^{12n-1} v^{\frac{j+1}{12}} \frac{j+1}{12} p_x + v^n {}_n p_x \\ &= v^{\frac{1}{12}} \sum_{j=0}^{12n-1} v^{\frac{j}{12}} \frac{j}{12} p_x - \sum_{j=1}^{12n} v^{\frac{j}{12}} \frac{j}{12} p_x + v^n {}_n p_x. \end{aligned}$$

The last term in the second sum is exactly $v^n {}_n p_x$, so that

$$\begin{aligned}
A_{x:n|}^{(12)} &= v^{\frac{1}{12}} \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}} p_x - \sum_{j=1}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}} p_x \\
&= v^{\frac{1}{12}} \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}} p_x - \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}} p_x + 1 \\
&= 1 - (1 - v^{\frac{1}{12}}) \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}} p_x.
\end{aligned}$$

Since, by part (a)

$$\sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}} p_x = 12 \ddot{a}_{x:n|}^{(12)},$$

and $12(1 - v^{\frac{1}{12}}) = d^{(12)}$, we arrive at the desired formula

$$A_{x:n|}^{(12)} = 1 - d^{(12)} \ddot{a}_{x:n|}^{(12)}.$$