MAS224, Actuarial Mathematics: Solutions to Problem Sheet 8.

Q1|5

M2 1*. Let £P be the monthly payment. Then P is determined by the equation of (expected) value $\boxed{100000 = 12P\ddot{a}^{(12)}_{[65]}}$.

M1 By making use of the approximation $\ddot{a}^{(p)}_x \approx \ddot{a}_x - \frac{p-1}{2p} = \frac{N_x}{D_x} - -\frac{p-1}{2p}$,

M1 table values

$$P = \frac{100000}{12 \times \ddot{a}^{(12)}_{[65]}} = \frac{100000}{12\left(\frac{N_{[65]}}{D_{[65]}} - \frac{11}{24}\right)} = \frac{100000}{12\left(\frac{22940.948}{2087.5676} - \frac{11}{24}\right)} = 791.32$$

A1 Thus £100,000 buys $\boxed{\text{£}791.32}$ per month paid in advance for life for a 65 old.

Q2|4

M1 2*. Compare the immediate prize of £10,000 with the expected present value of the monthly payments, which constitute a life annuity-due of £100 paid monthly (so £1200 per annum). The expected present value in pounds is just:

M1

M1 table values

$$\boxed{1200\ddot{a}^{(12)}_{60}} \approx 1200\left(\ddot{a}_{60} - \frac{11}{24}\right) = 1200\left(\frac{N_{60}}{D_{60}} - \frac{11}{24}\right) = 1200\left(\frac{35841.261}{2855.5942} - \frac{11}{24}\right) = 14511.49$$

A1 Since the expected present value of £100 per month is £14,511.49, which is greater than £10,000 it is preferable to take £100 per month.

Q3|6

M1 3*. (a) If the annual premium is £P then $\boxed{P\ddot{a}_{[50]} = 20{,}000A_{[50]}}$ — or $P = 20000\left(\frac{1}{\ddot{a}_{[50]}} - d\right)$, where $\ddot{a}_{[50]} = \frac{N_{[50]}}{D_{[50]}}$. By making use of the conversion relationship $A_x = 1 - d\ddot{a}_x$,

$$P = \frac{20{,}000 \times A_{[50]}}{\ddot{a}_{[50]}} = 20{,}000\left(\frac{D_{[50]}}{N_{[50]}} - d\right) = 20{,}000\left(\frac{4581.3224}{73544.823} - \frac{4}{104}\right) = 476.63$$

A1 The annual premium is $\boxed{\text{£}476.63.}$

(b) On what would have been his 61st birthday, the actual loss is just (the value at that time of the benefit actually paid) - (the value at that time of the payments he has actually made), i.e.

M1 $\boxed{20{,}000 - (1+i)^{11} \times P\ddot{a}_{\overline{11}|}}$ pounds, where $P = 476.63$ was the premium found in part (a). So the loss in pounds is — or, equiv., $\boxed{(1+i)^{11}(20000v^{11} - P\ddot{a}_{\overline{11}|})}$

$$\begin{aligned} 20{,}000 - (1+i)^{11} \times P\frac{1-v^{11}}{1-v} &= 20{,}000 - \frac{P(1+i)\left((1+i)^{11}-1\right)}{i} \\ &= 20{,}000 - 476.63\frac{(1.04)\left((1.04)^{11}-1\right)}{0.04} \\ &= \boxed{13{,}314.88} \end{aligned}$$

A1

(c) Ignoring expenses, the surrender value is the same as the policy value, and the latter is (the expected value, on John's 60th birthday, of the future benefit payment) - (the expected value, on John's 60th birthday, of the future premiums).

The expected value, on John's 60th birthday, of the future benefit payment (in pounds) is $20{,}000A_{[50]+10} = 20{,}000A_{60} = 20{,}000(1 - d\ddot{a}_{60})$.

The expected value, on John's 60th birthday, of the future premiums is (in pounds) $P\ddot{a}_{60}$, where P is the annual premium found in part (a).

We have $\ddot{a}_{60} = \frac{N_{60}}{D_{60}} = \frac{35841.261}{2855.5942}$.

Thus the surrender value of John's policy is

$$\begin{aligned} 20000(1-d\ddot{a}_{60}) - P\ddot{a}_{60} &= 20000\left(1-\frac{4}{104}\frac{35841.261}{2855.5942}\right) - 476.63\frac{35841.261}{2855.5942} \\ &= 4362.87 \quad \text{(to 2 d.p.)} \end{aligned}$$

M1 Alternatively, the surrender value of John's policy is $\boxed{£20,000 \times ({}_{10}V_{[50]})}$, where $\boxed{{}_mV_x = 1 - \ddot{a}_{x+m}/\ddot{a}_x}$, a result derived in lectures. This gives the following answer

A1

$$200000\left(1-\frac{\ddot{a}_{60}}{\ddot{a}_{[50]}}\right) = 200000\left(1-\frac{35841.261\times 4581.3224}{2855.5942\times 73544.823}\right) = \boxed{4362.92 \quad \text{(to 2 d.p.)}}$$

The discrepancy between two answers is due to the rounding off error in part (a).

Q4|10

4*. (a) Use the indicator-of-survival random variables I_j and follow the derivation of $\ddot{a}_{60}^{(p)}$ given in Lecture 25.

The P.V. of the annuity in question is $Z = \frac{1}{12}\sum_{j=0}^{12n-1} v^{j/12} I_j$, where I_j takes value 1 if (x) survives to age $x+\frac{j}{12}$ and 0 otherwise. The random variable I_j has Bernoulli distribution with parameter ${}_{\frac{j}{12}}p_x$. Thus

M1

$$\begin{aligned} \ddot{a}^{(12)}_{x:\overline{n}|} = E(Z) = \frac{1}{12}\sum_{j=0}^{12n-1} v^{\frac{j}{12}} E(I_j) &= \boxed{\frac{1}{12}\sum_{j=0}^{12n-1} v^{\frac{j}{12}}\,{}_{\frac{j}{12}}p_x} \qquad [\text{as} \quad E(I_j) = {}_{\frac{j}{12}}p_x] \\ &= \frac{1}{12}\sum_{j=0}^{12n-1} \frac{v^{x+\frac{j}{p}}}{v^x} \times \frac{l_{x+\frac{j}{p}}}{l_x} \\ &= \frac{1}{12}\sum_{k=0}^{12n-1} \frac{D_{x+\frac{j}{12}}}{D_x} \qquad [\text{as} \quad D_{x+t} = v^{x+t} l_{x+t}]. \end{aligned}$$

M1 derivation with explanations

(b) Follow the derivation of $A_x^{(p)}$ given in Lecture 25.

Let Z_1 and Z_2 be the present values of, correspondingly, the death and survival benefits in question. Then $A^{(12)}_{x:\overline{n}|} = E(Z_1) + E(Z_2)$.

M2 →

Z_1 is the present value of the death benefit in question. Z_1 takes values $v^{\frac{j+1}{12}}$, $j = 0, 1, \ldots, 12n-1$, with probabilities $P(\frac{j}{12} < T(x) < \frac{j+1}{12}) = {}_{\frac{j}{p}}p_x - {}_{\frac{j+1}{p}}p_x$, and it takes value 0 with probability $P(T(x) > n)$. It follows from this that

M1

$$E(Z_1) = \sum_{j=0}^{12n-1} v^{\frac{j+1}{12}} P\left(\frac{j}{12} < T(x) < \frac{j+1}{12}\right) = \boxed{\sum_{j=0}^{12n-1} v^{\frac{j+1}{12}}\left({}_{\frac{j}{12}}p_x - {}_{\frac{j+1}{12}}p_x\right)}.$$

M1 →

Z_2 takes value v^n if $P(T(x) > n)$ and 0 otherwise, so that it takes values v^n and 0 with probabilities ${}_np_x$ and $1 - {}_np_x$, correspondingly. It follows from this that

M1

$$\boxed{E(Z_2) = v^n \times ({}_np_x).}$$

Therefore

M1

$$\begin{aligned} \boxed{A^{(12)}_{x:\overline{n}|} = \sum_{j=0}^{12n-1} v^{\frac{j+1}{12}}\,{}_{\frac{j}{12}}p_x - \sum_{j=0}^{12n-1} v^{\frac{j+1}{12}}\,{}_{\frac{j+1}{12}}p_x + v^n\,{}_np_x} \\ = v^{\frac{1}{12}}\sum_{j=0}^{12n-1} v^{\frac{j}{12}}\,{}_{\frac{j}{12}}p_x - \sum_{j=1}^{12n} v^{\frac{j}{12}}\,{}_{\frac{j}{12}}p_x + v^n\,{}_np_x. \end{aligned}$$

The last term in the second sum is exactly $v^n {}_np_x$, so that

M2 derivation with explanations

$$\begin{aligned} A^{(12)}_{x:\overline{n}|} &= v^{\frac{1}{12}} \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}}p_x - \sum_{j=1}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}}p_x \\ &= v^{\frac{1}{12}} \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}}p_x - \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}}p_x + 1 \\ &= 1 - (1 - v^{\frac{1}{12}}) \sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}}p_x. \end{aligned}$$

Since, by part (a)

$$\sum_{j=0}^{12n-1} v^{\frac{j}{12}} {}_{\frac{j}{12}}p_x = 12\ddot{a}^{(12)}_{x:\overline{n}|},$$

and $12(1 - v^{\frac{1}{12}}) = d^{(12)}$, we arrive at the desired formula

$$A^{(12)}_{x:\overline{n}|} = 1 - d^{(12)}\,\ddot{a}^{(12)}_{x:\overline{n}|}.$$