MAS224, Actuarial Mathematics: Solutions to Problem Sheet 7.

1*. The annual premium is $£P$, where $P\ddot{a}_{[27]:\overline{23}|} = 50,000A_{[27]:\overline{23}|}$.

We use

$$A_{[27]:\overline{23}|} = 1 - d\ddot{a}_{[27]:\overline{23}|}, \quad \text{and} \quad \ddot{a}_{[27]:\overline{23}|} = \frac{N_{[27]} - N_{50}}{D_{[27]}},$$

so that

$$\begin{aligned} P &= \frac{50,000 \times A_{[27]:\overline{23}|}}{\ddot{a}_{[27]:\overline{23}|}} \\ &= 50,000\left(\frac{D_{[27]}}{N_{[27]} - N_{50}} - d\right) \\ &= 50,000\left(\frac{11755.233}{253646.27 - 73567.136} - \frac{4}{104}\right) = 1340.830654 \end{aligned}$$

Hence the annual premium is $£1,340.83$

If he is only able to pay a premium of $£500$ per annum, then he can purchase $£C$ of 23-year endowment assurance, where $500\ddot{a}_{[27]:\overline{23}|} = C \times A_{[27]:\overline{23}|}$. Therefore

$$C = \frac{500 \times \ddot{a}_{[27]:\overline{23}|}}{A_{[27]:\overline{23}|}} = \frac{500}{\left(\frac{D_{[27]}}{N_{[27]} - N_{50}} - d\right)} = \frac{500}{\left(\frac{11755.233}{253646.27 - 73567.136} - \frac{4}{104}\right)} = 18,645.16$$

£500 per annum paid in advance will buy £18,645.16 of 23-year endowment assurance for a life aged 27.

Alternatively: a premium of $£1340.830654$ buys $£50,000$ of endowment assurance, so a premium of $£500$ buys $£(500/1340.830654) \times 50,000 = £18,645.16$ of endowment assurance.

2*. The cost C (in pounds) of the endowment assurance in question is

$$C = 25000A_{[40]:\overset{}{\overline{25}|}^{\;1}} + 70000A^{1}_{[40]:\overline{25}|},$$

where $A_{[40]:\overline{25}|}^{\;\;\;\;1}$ and $A^{1}_{[40]:\overline{25}|}$ are, respectively, the expected present values of one unit of the survival benefit and one unit of the death benefit.

To work out the required annual premium P (in pounds), equate the cost of the endowment assurance to the cost (the expected present value) of the annuity of premiums. The latter is $P\ddot{a}_{[40]:\overline{25}|}$, so that

$$P = \frac{25000A_{[40]:\overline{25}|}^{\;\;\;\;1} + 70000A^{1}_{[40]:\overline{25}|}}{\ddot{a}_{[40]:\overline{25}|}}.$$

Recalling that

$$A_{[40]:\overline{25}|}^{\;\;\;\;1} = \frac{D_{[40]+25}}{D_{[40]}},\ A^{1}_{[40]:\overline{25}|} = 1 - \frac{D_{[40]+25}}{D_{[40]}} - d\frac{N_{[40]} - N_{[40]+25}}{D_{[40]}},\ \ddot{a}_{[40]:\overline{25}|} = \frac{N_{[40]} - N_{[40]+25}}{D_{[40]}},$$

we obtain the annual premium in terms of the monetary functions D and N:

$$\begin{aligned} P &= 25000\frac{D_{[40]+25}}{N_{[40]} - N_{[40]+25}} + 70000\left(\frac{D_{[40]}}{N_{[40]} - N_{[40]+25}} - \frac{D_{[40]+25}}{N_{[40]} - N_{[40]+25}} - d\right) \\ &= 70000\left(\frac{D_{[40]}}{N_{[40]} - N_{[40]+25}} - d\right) - 45000\frac{D_{[40]+25}}{N_{[40]} - N_{[40]+25}}. \end{aligned}$$

From the A1967-70 table,

$$\begin{aligned}
D_{[40]+25} &= D_{65} = 2144.1713 \\
N_{[40]+25} &= N_{65} = 23021.434 \\
D_{[40]} &= 6981.5977 \\
N_{[40]} &= 131995.19,
\end{aligned}$$

and

$$\begin{aligned}
P &= 70000\left(\frac{6981.5977}{131995.19 - 23021.434} - \frac{4}{104}\right) - 45000\frac{2144.1713}{131995.19 - 23021.434} \\
&= 906.95 \quad \text{(to 2 d.p.)}
\end{aligned}$$

Hence the annual premium is £906.95.

3*. (a) P.V. of the benefit payment

$$Z = \begin{cases} 6000v^{18}, & \text{if } T(0) > 18 \\ 0, & \text{otherwise} \end{cases}$$

where $T(0)$ is the exact future lifetime for a newborn and $v = \frac{1}{1+i} = \frac{1}{1.1}$.
Alternatively, $Z = 6000v^{18} \times \text{Bernoulli}(p)$ where $p = P(T(0) > 18) = {}_{18}p_0$.

$$\begin{aligned}
\text{E.P.V.} = E(Z) &= 6000v^{18} P(T(0) > 18) \\
&= 6000v^{18}\, {}_{18}p_0 = 6000v^{18}\frac{l_{18}}{l_0} = 6000\left(\frac{1}{1.1}\right)^{18}\frac{96514}{100000} = \underline{1041.53}
\end{aligned}$$

$\text{var}(Z) = (6000v^{18})^2({}_{18}p_0)({}_{18}q_0)$. Hence

$$\begin{aligned}
\text{st. dev.}(Z) = 6000v^{18}\sqrt{({}_{18}p_0)({}_{18}q_0)} &= 6000v^{18}\sqrt{\frac{l_{18}}{l_0}\left(1 - \frac{l_{18}}{l_0}\right)} \\
&= 6000\left(\frac{1}{1.1}\right)^{18}\sqrt{\frac{96514}{100000}\left(1 - \frac{96514}{100000}\right)} = \underline{197.94}
\end{aligned}$$

(b) Denote by Y the total present value of the benefit payments, $Y = \sum_{j=1}^{100} Z_j$, where Z_j is the present value of the benefit payment for boy j.
Then

$$\begin{aligned}
E(Y) &= 100 \times E(Y) = 100 \times 1041.53 = 104153 \\
\text{var}(Y) &= 100\text{var}(Y) \qquad \text{[as } Z_j \text{ are independent and have the same distribution]} \\
\text{st. dev.}(Y) &= 10 \times \text{st. dev.}(Y) = 10 \times 197.94 = 1979.4
\end{aligned}$$

The minimum amount is such value h that $P(Y \le h) = 0.95$.

$$P(Y \le h) = P\left(\frac{Y - E(Y)}{\sqrt{\text{var}(Y)}} \le \frac{h - E(Y)}{\sqrt{\text{var}(Y)}}\right) = 0.95$$

Normal approximation: $\frac{Y-E(Y)}{\sqrt{\text{var}(Y)}} \sim N(0,1)$. As $P(N(0,1) \leq 1.645) = 0.95$ we obtain an equation for h:

$$\frac{h - E(Y)}{\sqrt{\text{var}(Y)}} = 1.645$$

Hence,

$$h = E(Y) + 1.645 \times \text{st. dev.}(Y) = 104153 + 1.645 \times 1979.4 = \underline{107409.1} \quad \text{(1 d.p.)}$$

The minimum amount to be invested is £107409.1

4*. (a)

$$Y = \begin{cases} a_{\overline{K(x)|}} & \text{if } 0 \leq K(x) \leq n-1 \\ a_{\overline{n|}} & \text{if } K(x) \geq n \end{cases} \qquad \text{also accept} \quad Y = \begin{cases} a_{\overline{K(x)|}} & \text{if } 0 \leq K(x) \leq n \\ a_{\overline{n|}} & \text{if } K(x) \geq n+1 \end{cases}$$

(b)

$$Z = \begin{cases} v^{K(x)+1} & \text{if } 0 \leq K(x) < n+1 \\ Z = v^{n+1} & \text{if } K(x) \geq n+1 \end{cases}$$

Y can also be written as

$$Y = \begin{cases} a_{\overline{K(x)|}} & \text{if } 0 \leq K(x) \leq n \\ a_{\overline{n|}} & \text{if } K(x) \geq n+1 \end{cases},$$

making the comparison with Z easier.

If $0 \leq K(x) \leq n$ then $Z = v^{K(x)+1}$ and $Y = \frac{v(1-v^{K(x)})}{(1-v)} = \frac{v}{1-v} + \frac{v^{K(x)+1}}{1-v}$, so that $Y = \frac{v}{1-v} - \frac{Z}{1-v}$.

If $K(x) \geq n+1$ then $Z = v^{n+1}$ and $Y = \frac{v(1-v^n)}{(1-v)} = \frac{v}{1-v} + \frac{v^{n+1}}{1-v}$, so that $Y = \frac{v}{1-v} - \frac{Z}{1-v}$.

(c)

$$E[Y] = \frac{v}{1-v} - \frac{E[Z]}{1-v} = \frac{v}{1-v} - \frac{A_{x:\overline{n+1|}}}{1-v}$$

$$\text{var}(Y) = \frac{1}{(1-v)^2}\text{var}(Z) = \frac{1}{(1-v)^2}(E[Z^2] - (E[Z])^2) = \frac{1}{(1-v)^2}\left(A^*_{x:\overline{n+1|}} - (A_{x:\overline{n+1|}})^2\right)$$

where (as in lectures) * indicates that v is replaced by $v^* = v^2$ so that the interest rate is $i^* = (1+i)^2 - 1$.