MAS224, Actuarial Mathematics: Solutions to Problem Sheet 6.

1*. (a) The expected present value in pounds is $50,000A_{[28]}$, which is equal to

$$50,000\left(1-d\frac{N_{[28]}}{D_{[28]}}\right)=50,000\left(1-\frac{4}{104}\times\frac{241887.55}{11295.726}\right)=8,819.08 \quad \text{(2 d.p.)}.$$

The cost of assurance is £8,819.08.

(b) EITHER: Annual payments of £P in advance for life – have a whole-life annuity-due. Its expected present value in pounds is $P\,\ddot{a}_{[28]}$. Equating this to the cost of the life assurance, obtain an equation for P:

$$8,819.08=P\,\ddot{a}_{[28]},\quad \ddot{a}_{[28]}=\frac{N_{[28]}}{D_{[28]}}.$$

Hence

$$P=\frac{8,819.08}{\dfrac{N_{[28]}}{D_{[28]}}}=\frac{8,819.08}{\dfrac{241887.55}{11295.726}}=411.84 \quad \text{(2 d.p.)}.$$

£411.84 to be paid annually in advance for life.

OR: Use the formula $P=\frac{1}{\ddot{a}_{[28]}}-d$ obtained in lectures. Need to multiply 50,000.

$$P=50,000\left(\frac{1}{\ddot{a}_{[28]}}-d\right)=50,000\left(\frac{1}{\dfrac{241887.55}{11295.726}}-\frac{4}{104}\right)=411.84.$$

2*. Have a whole-life immediate annuity of £P per year. Its cost in pounds is $Pa_{[60]}$. By equating this to 50,000, obtain the annual payment $P=50,000/a_{[60]}$.

EITHER: $a_{[60]}=\frac{N_{[60]+1}}{D_{[60]}}$ $\quad N_{[60]+1}=32968.419$, $D_{[60]}=2815.3028$.

Note that we use $N_{[60]+1}$ and not $N_{[61]}$.

OR: $a_{[60]}=\ddot{a}_{[60]}-1=\frac{N_{[60]}}{D_{[60]}}-1$, $\quad N_{[60]}=35783.721$, $D_{[60]}=2815.3028$

$$P=\frac{50,000}{a_{[60]}}=\frac{50000}{\frac{N_{[60]+1}}{D_{[60]}}}=\frac{50000}{\frac{N_{[60]}}{D_{[60]}}-1}=4,269.70 \quad \text{(2 d.p.)}$$

John Doe will receive £4,269.70 annually.

3*. (a) $Z=v^m\ddot{a}_{\overline{k+1|}}$ if and only if $K(x)=m+k$ ($k+1$ payments to be made, the first one is in m years' time when the policyholder is age $x+m$)

$$P(Z=v^m\ddot{a}_{\overline{k+1|}})=P(K(x)=m+k)={}_{m+k}p_x-{}_{m+k+1}p_x.$$

(b) $E(Z)=\sum_{z_j=0}^{\infty}z_jP(Z=z_j)$. We can ignore zero value of Z as it gives no contribution to the expected value. It follows from part (a) that

$$\begin{aligned}E(Z)&=\sum_{k=0}^{\infty}v^m\ddot{a}_{\overline{k+1|}}\left({}_{m+k}p_x-{}_{m+k+1}p_x\right)\\&=\sum_{k=0}^{\infty}v^m\ddot{a}_{\overline{k+1|}}\left({}_{m+k}p_x\right)-\sum_{k=0}^{\infty}v^m\ddot{a}_{\overline{k+1|}}\left({}_{m+k+1}p_x\right).\end{aligned}$$

By making use of the we use the relation $\ddot{a}_{\overline{k+1}|} = \ddot{a}_{\overline{k}|} + v^k$ obtained in lectures,

$$\sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k+1}|} \left({}_{m+k}p_x\right) = \sum_{k=0}^{\infty} v^m \left(\ddot{a}_{\overline{k}|} + v^k\right) {}_{m+k}p_x = \sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k}|} \left({}_{m+k}p_x\right) + \sum_{k=0}^{\infty} v^k {}_{m+k}p_x$$

Changing the variable of summation from k to $k' = k+1$,

$$\sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k+1}|} \left({}_{m+k+1}p_x\right) = \sum_{k'=1}^{\infty} v^m \ddot{a}_{\overline{k'}|} \left({}_{m+k'}p_x\right).$$

Therefore

$$E(Z) = \sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k}|} \left({}_{m+k}p_x\right) + \sum_{k=0}^{\infty} v^{m+k} {}_{m+k}p_x - \sum_{k'=1}^{\infty} v^m \ddot{a}_{\overline{k'}|} \left({}_{m+k'}p_x\right).$$

Since $\ddot{a}_{\overline{0}|} = 0$, we have $\sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k}|} \left({}_{m+k}p_x\right) = \sum_{k=1}^{\infty} v^m \ddot{a}_{\overline{k}|} \left({}_{m+k}p_x\right)$. Therefore the first and third sums cancel each other, and

$$E(Z) = \sum_{k=0}^{\infty} v^{m+k} {}_{m+k}p_x = \sum_{k=0}^{\infty} v^{m+k} \frac{l_{x+k+m}}{l_x} = \sum_{k=0}^{\infty} \frac{v^{x+m+k} l_{x+k+m}}{v^x l_x} = \sum_{k=0}^{\infty} \frac{D_{x+m+k}}{D_x} = \frac{N_{x+m}}{D_x}.$$

4*. (a) $Z_1 = v^{T(x)}$ and p.d.f. of $T(x)$ is $f_{T(x)}(t) = -\frac{s'(x+t)}{s(x)}$.

$$\begin{aligned}
\bar{A}_x = E(Z_1) &= E(v^{T(x)}) = \int v^t f_{T(x)}(t)\,dt \\
&= -\frac{1}{s(x)} \int_0^{\infty} v^t s'(x+t)\,dt \\
&= -\frac{1}{s(x)} \sum_{k=0}^{\infty} \int_k^{k+1} v^t s'(x+t)\,dt \quad \text{[partitioning the interval of integration]} \\
&= -\frac{1}{s(x)} \sum_{k=0}^{\infty} \int_0^1 v^{\tau+k} s'(x+k+\tau)\,d\tau \quad \text{[substituting } \tau = t-k\text{]}.
\end{aligned}$$

(b) Linear interpolation: $s(x+k+t) \approx (1-t)s(x+k) + ts(x+k+1)$. Taking the derivative with respect to t, we obtain $s'(x+k+t) \approx s(x+k+1) - s(x+k)$. Therefore

$$-\frac{1}{s(x)} \sum_{k=0}^{\infty} \int_0^1 v^{t+k} s'(x+k+t)\,dt \quad \approx \quad \sum_{k=0}^{\infty} \frac{s(x+k) - s(x+k+1)}{s(x)} v^k \int_0^1 v^t\,dt.$$

Since $\frac{s(x+k)}{s(x)} = {}_kp_x$ and $\frac{s(x+k+1)}{s(x)} = {}_{k+1}p_x$, we conclude that

$$\bar{A}_x = -\frac{1}{s(x)} \sum_{k=0}^{\infty} \int_0^1 v^{t+k} s'(x+k+t)\,dt \approx \left(\int_0^1 v^t\,dt\right) \sum_{k=0}^{\infty} v^k \left({}_kp_x - {}_{k+1}p_x\right).$$

If δ is the force of mortality then $v = e^{-\delta}$ and

$$\int_0^1 v^t\,dt = \int_0^1 e^{-\delta t}\,dt = \frac{1-v}{\delta} = \frac{iv}{\delta}.$$

Therefore,

$$\bar{A}_x \approx \frac{i}{\delta} \sum_{k=0}^{\infty} v^{k+1} \left({}_kp_x - {}_{k+1}p_x\right).$$

It was shown in lectures that $A_x = E(Z_2) = \sum_{k=0}^{\infty} v^{k+1} \left({}_kp_x - {}_{k+1}p_x\right)$. Therefore $\bar{A}_x \approx \frac{i}{\delta} A_x$.