MAS224, Actuarial Mathematics: Solutions to Problem Sheet 6.

Q1 7

1*. (a) The expected present value in pounds is $\boxed{50,000A_{[28]}}$, which is equal to

M1

$$50,000\left(1-d\frac{N_{[28]}}{D_{[28]}}\right) = 50,000\left(1-\frac{4}{104}\times\frac{241887.55}{11295.726}\right) = 8,819.08 \quad \text{(2 d.p.)}.$$

M1 $A_{[28]} = 1 - d\frac{N_{[28]}}{D_{[28]}}$

The cost of assurance is £8,819.08.

M1 table values

A1

(b) EITHER: Annual payments of £P in advance for life – have a whole-life annuity-due. Its expected present value in pounds is $P\ddot{a}_{[28]}$. Equating this to the cost of the life assurance, obtain an equation for P:

M1 for either $8,819.08 = P\ddot{a}_{[28]}$ or $P = 50000\left(\frac{1}{\ddot{a}_{[28]}} - d\right)$

$$8,819.08 = P\ddot{a}_{[28]}, \quad \ddot{a}_{[28]} = \frac{N_{[28]}}{D_{[28]}}.$$

Hence

M1 for calculating $\ddot{a}_{[28]}$

$$P = \frac{8,819.08}{\frac{N_{[28]}}{D_{[28]}}} = \frac{8,819.08}{\frac{241887.55}{11295.726}} = 411.84 \quad \text{(2 d.p.)}.$$

A1

£411.84 to be paid annually in advance for life.

OR: Use the formula $P = \frac{1}{\ddot{a}_{[28]}} - d$ obtained in lectures. Need to multiply 50,000.

$$P = 50,000\left(\frac{1}{\ddot{a}_{[28]}} - d\right) = 50,000\left(\frac{1}{\frac{241887.55}{11295.726}} - \frac{4}{104}\right) = 411.84.$$

Q2 4

2*. Have a whole-life immediate annuity of £P per year. Its cost in pounds is $Pa_{[60]}$. By equating this to 50,000, obtain the annual payment $P = 50,000/a_{[60]}$.

M2 for equation of value $50000 = Pa_{[60]}$

EITHER: $a_{[60]} = \frac{N_{[60]+1}}{D_{[60]}}$ $\quad N_{[60]+1} = 32968.419$, $D_{[60]} = 2815.3028$.

Note that we use $N_{[60]+1}$ and not $N_{[61]}$.

OR: $a_{[60]} = \ddot{a}_{[60]} - 1 = \frac{N_{[60]}}{D_{[60]}} - 1$, $\quad N_{[60]} = 35783.721$, $D_{[60]} = 2815.3028$

M1 for calculation of $a_{[60]}$

$$P = \frac{50,000}{a_{[60]}} = \frac{50000}{\frac{N_{[60]+1}}{D_{[60]}}} = \frac{50000}{\frac{N_{[60]}}{D_{[60]}} - 1} = 4,269.70 \quad \text{(2 d.p.)}$$

A1

John Doe will receive £4,269.70 annually.

Q3 7

3*. (a) $\boxed{Z = v^m\ddot{a}_{\overline{k+1}|} \text{ if and only if } K(x) = m+k}$ ($k+1$ payments to be made, the first one is in m years' time when the policyholder is age $x+m$)

A2

A1

$$\boxed{P(Z = v^m\ddot{a}_{\overline{k+1}|}) = P(K(x) = m+k) = {}_{m+k}p_x - {}_{m+k+1}p_x.}$$

(b) $E(Z) = \sum_{z_j=0}^{\infty} z_j P(Z = z_j)$. We can ignore zero value of Z as it gives no contribution to the expected value. It follows from part (a) that

M1 for stating this

$$\boxed{E(Z) = \sum_{k=0}^{\infty} v^m\ddot{a}_{\overline{k+1}|}\left({}_{m+k}p_x - {}_{m+k+1}p_x\right)}$$

$$= \sum_{k=0}^{\infty} v^m\ddot{a}_{\overline{k+1}|}\left({}_{m+k}p_x\right) - \sum_{k=0}^{\infty} v^m\ddot{a}_{\overline{k+1}|}\left({}_{m+k+1}p_x\right).$$

M3 for derivation

By making use of the we use the relation $\ddot{a}_{\overline{k+1}|} = \ddot{a}_{\overline{k}|} + v^k$ obtained in lectures,

$$\sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k+1}|}\left({}_{m+k}p_x\right) = \sum_{k=0}^{\infty} v^m \left(\ddot{a}_{\overline{k}|} + v^k\right){}_{m+k}p_x = \sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k}|}\left({}_{m+k}p_x\right) + \sum_{k=0}^{\infty} v^k{}_{m+k}p_x$$

Changing the variable of summation from k to $k' = k+1$,

$$\sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k+1}|}\left({}_{m+k+1}p_x\right) = \sum_{k'=1}^{\infty} v^m \ddot{a}_{\overline{k'}|}\left({}_{m+k'}p_x\right).$$

Therefore

$$E(Z) = \sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k}|}\left({}_{m+k}p_x\right) + \sum_{k=0}^{\infty} v^{m+k}{}_{m+k}p_x - \sum_{k'=1}^{\infty} v^m \ddot{a}_{\overline{k'}|}\left({}_{m+k'}p_x\right).$$

Since $\ddot{a}_{\overline{0}|} = 0$, we have $\sum_{k=0}^{\infty} v^m \ddot{a}_{\overline{k}|}\left({}_{m+k}p_x\right) = \sum_{k=1}^{\infty} v^m \ddot{a}_{\overline{k}|}\left({}_{m+k}p_x\right)$. Therefore the first and third sums cancel each other, and

$$E(Z) = \sum_{k=0}^{\infty} v^{m+k}{}_{m+k}p_x = \sum_{k=0}^{\infty} v^{m+k}\frac{l_{x+k+m}}{l_x} = \sum_{k=0}^{\infty} \frac{v^{x+m+k}l_{x+k+m}}{v^x l_x} = \sum_{k=0}^{\infty} \frac{D_{x+m+k}}{D_x} = \frac{N_{x+m}}{D_x}.$$

Q4

A1 if both correct

4*. (a) $\boxed{Z_1 = v^{T(x)}}$ and p.d.f. of $T(x)$ is $\boxed{f_{T(x)}(t) = -\frac{s'(x+t)}{s(x)}}$.

M1

$$\begin{aligned}
\bar{A}_x = E(Z_1) &= E(v^{T(x)}) = \int v^t f_{T(x)}(t)\,dt \\
&= -\frac{1}{s(x)}\int_0^{\infty} v^t s'(x+t)\,dt \\
&= -\frac{1}{s(x)}\sum_{k=0}^{\infty}\int_k^{k+1} v^t s'(x+t)\,dt \quad \text{[partitioning the interval of integration]} \\
&= -\frac{1}{s(x)}\sum_{k=0}^{\infty}\int_0^{1} v^{\tau+k} s'(x+k+\tau)\,d\tau \quad [\text{substituting } \tau = t-k].
\end{aligned}$$

M1

M1

(b) Linear interpolation: $s(x+k+t) \approx (1-t)s(x+k) + ts(x+k+1)$. Taking the derivative with respect to t, we obtain $\boxed{s'(x+k+t) \approx s(x+k+1) - s(x+k)}$. Therefore

$$-\frac{1}{s(x)}\sum_{k=0}^{\infty}\int_0^1 v^{t+k}s'(x+k+t)\,dt \approx \sum_{k=0}^{\infty}\frac{s(x+k)-s(x+k+1)}{s(x)}v^k\int_0^1 v^t\,dt.$$

M1

Since $\frac{s(x+k)}{s(x)} = {}_kp_x$ and $\frac{s(x+k+1)}{s(x)} = {}_{k+1}p_x$, we conclude that

$$\bar{A}_x = -\frac{1}{s(x)}\sum_{k=0}^{\infty}\int_0^1 v^{t+k}s'(x+k+t)\,dt \approx \left(\int_0^1 v^t\,dt\right)\sum_{k=0}^{\infty} v^k\left({}_kp_x - {}_{k+1}p_x\right).$$

If δ is the force of mortality then $v = e^{-\delta}$ and

M1 Evaluation of integral

$$\int_0^1 v^t\,dt = \int_0^1 e^{-\delta t}\,dt = \frac{1-v}{\delta} = \frac{iv}{\delta}.$$

Therefore,

M1 for recognising $A_x = \sum v^{k+1}({}_kp_x - {}_{k+1}p_x)$

$$\bar{A}_x \approx \frac{i}{\delta}\sum_{k=0}^{\infty} v^{k+1}\left({}_kp_x - {}_{k+1}p_x\right).$$

It was shown in lectures that $A_x = E(Z_2) = \sum_{k=0}^{\infty} v^{k+1}\left({}_kp_x - {}_{k+1}p_x\right)$. Therefore $\bar{A}_x \approx \frac{i}{\delta}A_x$.