MAS224, Actuarial Mathematics: Solutions to Problem Sheet 5.

1*. The assumption of the uniform distribution of deaths within each year of life implies that $l_{x+t} = (1-t)l_x + tl_{x+1}$ for integer x and $0 < t < 1$ (a result obtained in Lecture 15).

Then

◇ $\quad {}_tp_x = \frac{l_{x+t}}{l_x} = \frac{(1-t)l_x+tl_{x+1}}{l_x} = 1 - t + t(p_x).$

◇ $\quad {}_{1-t}p_{x+t} = \frac{l_{x+t+(1-t)}}{l_{x+t}} = \frac{l_{x+1}}{(1-t)l_x+tl_{x+1}} = \frac{l_{x+1}}{1-t(l_x-l_{x+1})}$

Dividing numerator and denominator by l_x gives ${}_{1-t}p_{x+t} = \frac{p_x}{1-t(q_x)}$.

2*. (a) Want ${}_{\frac{1}{4}}q_{18}$. Use formula ${}_tq_x = t(q_x)$ obtained in lectures.

◇ $\quad {}_{\frac{1}{4}}q_{18} = (\frac{1}{4})(q_{18}) = (\frac{1}{4})(0.00112) = 0.00028.$

(b) Want ${}_{\frac{1}{2}}p_{50}$. Use formula ${}_tp_x = 1 - t + t(p_x)$ obtained in Problem 1.

◇ $\quad {}_{\frac{1}{2}}p_{50} = 1 - (\frac{1}{2}) + (\frac{1}{2})p_{50} = 0.5 + 0.5 \times (0.99272) = 0.99636 = 0.9964$ (to 4 s.f.)

Alternatively, ${}_{\frac{1}{2}}p_{50} = 1 - {}_{\frac{1}{2}}q_{50} = 1 - (\frac{1}{2})(q_{50}) = 1 - (\frac{1}{2})(0.00728) = 0.99636.$

(c) Want ${}_{4\frac{11}{12}|\frac{1}{12}}q_{60}$. Use linear interpolation $l_{x+t} = (1-t)l_x + tl_{x+1}$ in the usual way.

$$\begin{aligned}
{}_{4\frac{11}{12}|\frac{1}{12}}q_{60} &= \frac{l_{64\frac{11}{12}} - l_{65}}{l_{60}} \\
&= \frac{(\frac{1}{12})l_{64} + (\frac{11}{12})l_{65} - l_{65}}{l_{60}} \\
&= \frac{(\frac{1}{12})(l_{64} - l_{65})}{l_{60}} \\
&= \frac{(\frac{1}{12})(d_{64})}{l_{60}} \\
&= \frac{2366}{(12)(78924)} = 0.002498 \text{ (to 4 s.f.)}
\end{aligned}$$

(e) From $l_0 = 100,000$ newborns, the expected number who die within 9 months (either way) of their 21st birthday would be ${}_{1\frac{1}{2}}d_{20\frac{1}{4}}$, so just scale this by $1000/100000 = 0.01$. Use linear interpolation $l_{x+t} = (1-t)l_x + tl_{x+1}$ in the usual way.

$$\begin{aligned}
0.01 \times {}_{1\frac{1}{2}}d_{20\frac{1}{4}} &= 0.01(l_{20\frac{1}{4}} - l_{21\frac{3}{4}}) \\
&= 0.01\left(\left(\frac{3}{4}l_{20} + \frac{1}{4}l_{21}\right) - \left(\frac{1}{4}l_{21} + \frac{3}{4}l_{22}\right)\right) \\
&= 0.01\left(\frac{3}{4}(l_{20} - l_{22})\right) \\
&= \frac{0.03(96293 - 96065)}{4} = 1.71
\end{aligned}$$

Alternatively use result that number dying has binomial distribution with parameters 1000 and ${}_{20\frac{1}{4}|1\frac{1}{2}}q_0$ so the expected number dying is $1000({}_{20\frac{1}{4}|1\frac{1}{2}}q_0) = 1000\frac{l_{20\frac{1}{4}} - l_{21\frac{3}{4}}}{l_0}$ which gives the same result as above.

3*. $n_x = Al_x$ and $N(x) = Al_x(0.5 + \mathring{e}_x)$, where $N(x)$ is the expected number aged at least x in the population.

(i) $N(0)$ is the expected population size. Therefore $5,000 = N(0) = Al_0(0.5 + \mathring{e}_0)$. Hence $A = \frac{5000}{100,000(0.5+68.09)} = 0.00072896924$ (The value of A given can just be to 4 significant digits, but you should use more digits for calculations in (ii) and (iii))

(ii) The expected number of births each year is just the expected number of newborns n_0.

$$n_0 = Al_0 = 0.00072896924 \times 100,000 = 72.8969$$

(iii) The expected number in the population who are aged under 60 is just $N(0) - N(60) = 5000 - Al_{60}(0.5 + \mathring{e}_{60})$.

$$N(0) - N(60) = 5000 - 0.00072896924 \times 78924(0.5 + 15.06) = 4,104.7842$$

4*. (a)

$$_{34}p_{[26]} = \frac{l_{[26]+34}}{l_{[26]}} = \frac{l_{60}}{l_{[26]}} = \frac{30039.787}{33917.341} = 0.8857$$

(b)

$$_4q_{[65]+1} = \frac{l_{[65]+1} - l_{[65]+5}}{l_{65]+1}} = \frac{l_{[65]+1} - l_{70}}{l_{[65]+1}} = \frac{26459.331 - 23622.102}{26459.331} = 0.1072$$

(c)

$$_{9|1}q_{[65]} = \frac{l_{[65]+9} - l_{[65]+10}}{l_{[65]}} = \frac{l_{74} - l_{75}}{l_{[65]}} = \frac{19623.545 - 18507.942}{26718.225} = 0.04175$$

(d) The expected number who die within a year of retirement is just $100 \times {}_{37|1}q_{[23]}$

$$100 \times {}_{37|1}q_{[23]} = 100\frac{l_{60} - l_{61}}{l_{[23]}} = \frac{100 \times (30039.787 - 29606.239)}{33990.921} = 1.2786$$

(e) The probability that he only makes the first four payments is just the probability that his further lifetime is greater than 3 months but less than 4 months. So we want to find ${}_{\frac{1}{4}|\frac{1}{12}}q_{[30]}$. We use linear interpolation in the usual way.

$$\begin{aligned}
{}_{\frac{1}{4}|\frac{1}{12}}q_{[30]} &= \frac{l_{[30]+\frac{1}{4}} - l_{[30]+1/3}}{l_{[30]}} \\
&= \frac{\left((\frac{3}{4})l_{[30]} + (\frac{1}{4})l_{[30]+1}\right) - \left((\frac{2}{3})l_{[30]} + (\frac{1}{3})l_{[30]+1}\right)}{l_{[30]}} \\
&= \frac{(\frac{1}{12})l_{[30]} - (\frac{1}{12})l_{[30]+1}}{l_{[30]}} \\
&= \frac{33829.764 - 33813.958}{12 \times 33829.764} = 0.00003894
\end{aligned}$$

5. To calculate the table we need to use:

$$
\begin{aligned}
l_{[x]+1} &= \frac{l_{x+2}}{p_{[x]+1}} = \frac{l_{x+2}}{0.9 \times p_{x+1}} = \frac{l_{x+1}}{0.9} \\
l_{[x]} &= \frac{l_{[x]+1}}{p_{[x]}} = \frac{l_{[x]+1}}{0.7 \times p_x} = \frac{l_{x+1}}{0.9 \times 0.7 \times p_x} = \frac{l_x}{0.63}
\end{aligned}
$$

Hence

- ◇ $l_{[63]} = l_{63}/0.63 = 73,084/0.63 = 116,006.3492$
- ◇ $l_{[64]} = l_{64}/0.63 = 70856/0.63 = 112,469.8413$
- ◇ $l_{[65]} = l_{65}/0.63 = 68490/0.63 = 108,714.2857$
- ◇ $l_{[63]+1} = l_{64}/0.9 = 70856/0.9 = 78,728.8889$
- ◇ $l_{[64]+1} = l_{65}/0.9 = 68490/0.9 = 76,100$
- ◇ $l_{[65]+1} = l_{66}/0.9 = 65991/0.9 = 73,323.3333$

So the mini version of the table with entries the select mortality functions $l_{[x]},\ l_{[x]+1}$ and l_{x+2} for $x = 63, 64, 65$ based on ELT12 is

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
63	116,006.3492	78,728.8889	68,490	65
64	112,469.8413	76,100	65,991	66
65	108,714.2857	73,323.3333	63,366	67

The expressions for $\mathring{e}_{[x]}$ and $\mathring{e}_{[x]+1}$ in this example become:

$$
\begin{aligned}
\mathring{e}_{[x]} &= \frac{1}{2} + \frac{1}{l_{[x]}}\left(l_{[x]+1} + l_{x+1}\left(\mathring{e}_{x+1} - \frac{1}{2}\right)\right) \\
&= \frac{1}{2} + \frac{0.63}{l_x}\left(\frac{l_{x+1}}{0.9} + l_{x+1}\left(\mathring{e}_{x+1} - \frac{1}{2}\right)\right) \\
&= \frac{1}{2} + \frac{l_{x+1}}{l_x}\left(0.7 + 0.63\left(\mathring{e}_{x+1} - \frac{1}{2}\right)\right) \\
&= \frac{1}{2} + p_x\left(0.7 + 0.63\left(\mathring{e}_{x+1} - \frac{1}{2}\right)\right)
\end{aligned}
$$

$$
\mathring{e}_{[x]+1} = \frac{1}{2} + \frac{l_{x+1}}{l_{[x]+1}}\left(\mathring{e}_{x+1} - \frac{1}{2}\right) = \frac{1}{2} + 0.9\left(\mathring{e}_{x+1} - \frac{1}{2}\right)
$$

Therefore

$$
\begin{aligned}
\mathring{e}_{[63]} &= 0.5 + 0.96951(0.7 + 0.63(12.54 - 0.5)) = 8.5326 \\
\mathring{e}_{[63]+1} &= 0.5 + 0.9(12.54 - 0.5) = 11.336 \\
\mathring{e}_{[64]} &= 0.5 + 0.96661(0.7 + 0.63(11.95 - 0.5)) = 8.1493.
\end{aligned}
$$