MAS224, Actuarial Mathematics: Solutions to Problem Sheet 4.

1*. Calculations for the table use:

- ◇ $l_0 = 1,000$ and $l_{x+1} = l_x p_x$ for $x = 0, 1, 2, 3, 4$.
- ◇ $d_x = l_x - l_{x+1}$ for $x = 0, 1, 2, 3, 4$.
- ◇ $q_x = 1 - p_x$ for $x = 0, 1, 2, 3, 4$.
- ◇ $e_x = (l_{x+1} + ... + l_4)/l_x$ and $\mathring{e}_x \approx e_x + 0.5$ for $x = 0, 1, 2, 3$. Note that a beetle age 4 can't survive a whole further year so $e_4 = 0$ and $\mathring{e}_x = 0.5$.

x	l_x	d_x	p_x	q_x	$\mathring{e}_x$	x
0	1000	100	0.9	0.1	1.949	0
1	900	450	0.5	0.5	1.111	1
2	450	360	0.2	0.8	0.72	2
3	90	81	0.1	0.9	0.6	3
4	9	9	0	1	0.5	4

- ◇ ${}_3p_1 = \frac{l_4}{l_1} = \frac{9}{900} = 0.01$
- ◇ ${}_2q_2 = \frac{l_2 - l_4}{l_4} = \frac{450-9}{450} = 0.98$
- ◇ ${}_{2|1}q_1 = \frac{l_3 - l_4}{l_1} = \frac{90-9}{900} = 0.09$
- ◇ $K(1)$ takes values 0,1,2,3 since a beetle aged 1 is dead by age 5 (and so can at most live 3 complete further years). Now $P(K(1) = k) = {}_kp_1 - {}_{k+1}p_1 = \frac{l_{k+1} - l_{k+2}}{l_1}$, so

  $P(K(1) = 0) = \frac{l_1 - l_2}{l_1} = \frac{450}{900} = 0.5,$
  $P(K(1) = 1) = \frac{l_2 - l_3}{l_1} = \frac{360}{900} = 0.4,$
  $P(K(1) = 2) = \frac{l_3 - l_4}{l_1} = \frac{81}{900} = 0.09,$
  $P(K(1) = 3) = \frac{l_4 - l_5}{l_1} = \frac{9}{900} = 0.01$

2*. (a) P(man aged 60 dies during the next five years) is just ${}_5q_{60} = \frac{l_{60} - l_{65}}{l_{60}} = \frac{78924-68490}{78924} = 0.1322$

(b) P(man aged 40 dies by age 65) is just ${}_{25}q_{40} = \frac{l_{40} - l_{65}}{l_{40}} = \frac{93790-68490}{93790} = 0.2698$

(c) P(man aged 20 survives to age 50 but dies before age 60) is just ${}_{30|10}q_{20} = \frac{l_{50} - l_{60}}{l_{20}} = \frac{90085-78924}{96293} = 0.1159$

(d) The expected number of deaths of men within 1 year of retirement at age 65 out of 100,000 newborns is just $d_{65} = 2499$

Alternatively the probability that a newborn dies between 65 and 66 is $S(65) - S(66) = (l_{65} - l_{66})/l_0 = d_{65}/100,000$. The number dying has binomial distribution with n parameter 100,000 and this probability as p parameter. So expected number is $100,000 d_{65}/100,000$ which is same result as above.

(e) The expected number alive at age 30 out of 100,000 newborns is just l_{30}, so the expected number dead by age 30 is just $100,000 - l_{30}$. So if it is out of 1,000 you just scale by $1,000/100,000 = 0.01$ giving $0.01(100,000 - 95265) = 47.35$

Alternatively, the number of men who die by age 30 out of 1,000 newborns has binomial distribution with parameters 1,000 and ${}_{30}q_0$. So the expected number is just $1,000 {}_{30}q_0 = 1,000\frac{l_0 - l_{30}}{l_0} = \frac{1,000(100,000-95265)}{100,000} = 47.35$

(f) The number of men who will survive to age 65 out of 1,000 who are aged 50 now has binomial distribution parameters 1000 and ${}_{15}p_{50}$, so the expected number is just $1000 {}_{15}p_{50} = 1000\frac{l_{65}}{l_{50}} = \frac{1000(68490)}{90085} = 760.2820$

3. $S(x) = 1 - \frac{x^2}{36}$ for $0 \le x \le 6$.

(a) $p_x = \frac{S(x+1)}{S(x)} = \frac{36-(x+1)^2}{36-x^2}$, $q_x = 1 - p_x$, $l_x = l_0 S(x) = 100(36 - x^2)$ and $d_x = l_x - l_{x+1}$. Also $\mathsf{e}_x = \frac{1}{l_x}(l_{x+1} + ... + l_5)$ for $x = 0, ..., 4$.

Year	p_x	q_x	l_x	d_x	e_x
0	35/36	1/36	3,600	100	12500/3600=3.4722
1	32/35	3/35	3,500	300	9000/3500=2.5714
2	27/32	5/32	3,200	500	5800/3200=1.8125
3	20/27	7/27	2700	700	3100/2700=1.1481
4	11/20	9/20	2000	900	1100/2000=0.55
5	0	1	1100	1100	0

(b) $\mu(x) = \frac{-S'(x)}{S(x)} = \frac{2x}{36-x^2}$ for $0 < x < 6$.

(c) For $0 \le x < 6$,

$$\begin{aligned} \mathring{\mathsf{e}}_x &= \tfrac{1}{36-x^2} \int_0^{6-x} (36 - (x+t)^2) dt \\ &= \tfrac{1}{36-x^2} \left[36t - \tfrac{(x+t)^3}{3} \right]_{t=0}^{t=6-x} \\ &= \tfrac{36(6-x)-(1/3)(6^3-x^3)}{36-x^2} \\ &= \tfrac{(6-x)(108-(x^2+6x+36))}{3(36-x^2)} \\ &= \tfrac{72-6x-x^2}{3(6+x)} \\ &= \tfrac{(6-x)(12+x)}{3(6+x)} \end{aligned}$$

4*.

$$\mathsf{e}_x = \frac{1}{l_x} \sum_{r=1}^{\infty} l_{x+r} = \frac{l_{x+1} + l_{x+2} + ...}{l_x}$$

$$1 + \mathsf{e}_{x+1} = 1 + \frac{1}{l_{x+1}} \sum_{r=1}^{\infty} l_{x+1+r} = \frac{1}{l_{x+1}} \left(l_{x+1} + \sum_{r=1}^{\infty} l_{x+r+1} \right) = \frac{l_{x+1} + l_{x+2} + l_{x+3} + ...}{l_{x+1}}$$

So $\mathsf{e}_x = \frac{l_{x+1}}{l_x}(1 + \mathsf{e}_{x+1}) = p_x(1 + \mathsf{e}_{x+1})$. The result that $\mathsf{e}_x \le (1 + \mathsf{e}_{x+1})$ then immediately follows since $p_x \le 1$.

Thus $\mathsf{e}_x = 1 + \mathsf{e}_{x+1}$ when $p_x = 1$, i.e. if a life aged x is certain to survive the next year.

5*. For the survival function we have $s(x) = e^{-\int_0^x \mu(u)du}$.

If $\mu(x) = 2/(50 - x)$ then $-\int_0^x \mu(u)du = 2\ln(50 - x) - 2\ln(50)$ for $0 \le x < 50$. Hence $s(x) = \left(\frac{50-x}{50}\right)^2$ for $0 \le x < 50$. Since $s(x)$ is a decreasing function of x, and $s(x)$ tends to 0 as x tends to 50, $s(x) = 0$ for $x = 50$ and beyond.

The p.d.f. of $T(x)$ is given by the formula

$$f_{T(x)}(t) = -\frac{s'(x+t)}{s(x)}.$$

If $s(x) = \left(\frac{50-x}{50}\right)^2$ for $0 \le x < 50$ then $s'(x) = -\frac{2}{50}\left(\frac{50-x}{50}\right)$,

$$s'(x+t) = -\frac{2}{50}\left(\frac{50-x-t}{50}\right), 0 \le t < 50,$$

and

$$f_{T(x)}(t) = \frac{2(50-x-t)}{(50-x)^2}, \quad 0 \le t < 50-x.$$

$f_{T(x)}(t)$ vanishes for $t > 50 - x$ because no individual lives beyond 50 years.

We have two formulae to choose from for the evaluation of $\mathring{e}_x = E(T(x))$:

$$\mathring{e}_x = \int_0^\infty t f_{T(x)}(t)dt \quad \text{or} \quad \mathring{e}_x = \frac{1}{s(x)} \int_0^\infty s(x+t)dt$$

The second one involves fewer calculations. Note that there is a cut off in the integral at $t = 30$, as $s(20+t) = 0$ for $t > 30$.

$$\mathring{e}_{20} = \frac{1}{s(20)} \int_0^{30} s(20+t)dt = \frac{1}{(30)^2} \int_0^{30} (30-t)^2 dt = \frac{1}{(30)^2} \frac{(30)^3}{3} = 10.$$

Thus $\mathring{e}_{20} = 10$, i.e. the complete expectation of further life at age 20 is 10 years.