MAS224, Actuarial Mathematics: Solutions to Problem Sheet 3.

1*. For John Doe we are given that the interest rate is 0.5% per month, meaning $i^{(12)}/12 = 0.005$. The total amount paid per annum is £1200. So the P.V. of his investments is (in pounds) $1200\ddot{a}_{\overline{10}|}^{(12)}$, and you obtain the accumulated value £A of his investments in 10 years time by scaling the P.V. by $(1+i)^{10}$:

$$A = 1200\ddot{a}_{\overline{10}|}^{(12)} \times (1+i)^{10} = \frac{1200}{12}\frac{1-v^{10}}{1-v^{1/12}} \times (1+i)^{10}.$$

Here i is the effective annual rate and $v = 1/(1+i)$. To work out the effective annual rate and the discounting factor v from the monthly rate, we can use the relation

$$1+i = \left(1+\frac{i^{(12)}}{12}\right)^{12}, \qquad \frac{i^{(12)}}{12} = 0.005.$$

Thus, $1+i = (1.005)^{12}$, $(1+i)^{10} = (1.005)^{120}$, and

$$v = \left(\frac{1}{1+\frac{i^{(12)}}{12}}\right)^{12} = \left(\frac{1}{1.005}\right)^{12}.$$

Hence

$$A = 100\frac{1-\left(\frac{1}{1.005}\right)^{120}}{1-\frac{1}{1.005}} \times (1.005)^{120} = 100\frac{(1.005)^{120}-1}{1-1/1.005} = 16469.8744 \quad \text{(4dp)}$$

So John Doe's investments grow to £$16,469.87$.

For Jane Roe, we split her investments into 2 parts - the first 3 years of money invested and the remaining 7 years.

First 3 years monthly investments have present value (in pounds) $1200\ddot{a}_{\overline{3}|}^{(12)}$, where $i = 0.07$. To get the value in 3 years we scale this by $(1.07)^3$ and to get the value in 10 years time we scale again by $(1.06)^7$.

So the value in pounds is

$$1200 \times \frac{1-(1/1.07)^3}{12(1-(1/1.07)^{1/12})} \times (1+0.07)^3 \times (1+0.06)^7 = 6018.499693.$$

The remaining 7 years monthly investments have value at the start of the 7 years (i.e. in 3 years time) of $1200\ddot{a}_{\overline{7}|}^{(12)}$, where $i = 0.06$. So we scale this by $(1.06)^7$ to get the value 10 years from now.

So the value in pounds is

$$1200 \times \frac{1-(1/1.06)^7}{12\left(1-(1/1.06)^{1/12}\right)} \times (1.06)^7 = 10397.0508$$

Adding the two values together, 6018.499693+10397.0508=16415.5505 (4 dp). Hence the total amount Jane Roe's investment grows to in 10 years is £$16,415.55$.

2*. (a*) Let the monthly repayment be £P, so the annual amount is £12P. Then $25,000 = 12Pa^{(12)}_{\overline{20}|}$. Hence

$$P = \frac{25000}{12} \times \frac{12\left(1-\left(\frac{1}{1.1}\right)^{1/12}\right)}{\left(\frac{1}{1.1}\right)^{1/12}\left(1-\left(\frac{1}{1.1}\right)^{20}\right)} = \frac{25000\left((1.1)^{1/12}-1\right)}{\left(1-\left(\frac{1}{1.1}\right)^{20}\right)} = 234.1598854$$

Hence the monthly repayment is £234.16

(b) Find n such that $25000 = 12 \times 300a^{(12)}_{\overline{n}|}$ and convert to months. Round <u>up</u> to years and whole months to n^*, where n^* is expressed in years so is of the form $\frac{integer}{12}$. Then

$$12 \times 300a^{(12)}_{\overline{n^*-(1/12)}|} < 25000 < 12 \times 300a^{(12)}_{\overline{n^*}|}$$

Then n^* is the time until the loan is repaid. We now find this value.

$$25000 = 12 \times 300\frac{\left(\frac{1}{1.1}\right)^{1/12}\left(1-\left(\frac{1}{1.1}\right)^{n}\right)}{12\left(1-\left(\frac{1}{1.1}\right)^{1/12}\right)} = \frac{300\left(1-\left(\frac{1}{1.1}\right)^{n}\right)}{((1.1)^{1/12}-1)}$$

so re-arranging we obtain

$$n = \frac{\ln\left(1-\frac{25000}{300}\left((1.1)^{1/12}-1\right)\right)}{-ln(1.1)} = 11.459$$

Since $5 < 12 \times 0.459 < 6$, n^* is 11 years and 6 months, i.e. eleven and a half years. This is the time until the loan is repaid.

So £300 is paid for eleven years and 5 months (i.e. 137 months). We now calculate the final repayment at eleven years and six months. Let the last payment be £C, then this is the outstanding amount at time $n^* = 11.5$. So it is the value at time n^* of the loan minus the value at the same time of the previous payments, i.e.

$$\begin{aligned} C &= (1+i)^{n^*}\left(25000 - 12 \times 300a^{(12)}_{\overline{n^*-(1/12)}|}\right) \\ &= (1.1)^{11.5}\left(25000 - \frac{300 \times 12\left(\frac{1}{1.1}\right)^{1/12}\left(1-\left(\frac{1}{1.1}\right)^{137/12}\right)}{12\left(1-\left(\frac{1}{1.1}\right)^{1/12}\right)}\right) \\ &= 153.03 \end{aligned}$$

Therefore the last payment is £153.03

3*. (a) Let the annual payment for loan 1 be $£P_1$, then $P_1a_{\overline{10}|} = 5000$. So

$$P_1 = \frac{5000\left(1-\frac{1}{1.1}\right)}{\left(\frac{1}{1.1}\right)\left(1-\left(\frac{1}{1.1}\right)^{10}\right)} = 813.73$$

Let the annual payment for loan 2 be $£P_2$, then $P_2a_{\overline{10}|} = 10000$. So

$$P_2 = \frac{10000\left(1-\frac{1}{1.12}\right)}{\left(\frac{1}{1.12}\right)\left(1-\left(\frac{1}{1.12}\right)^{10}\right)} = 1769.84$$

Let the annual payment for loan 3 be $\pounds P_3$, then $P_3 a_{\overline{3}|} = 3000$. So

$$P_3 = \frac{3000\left(1-\frac{1}{1.15}\right)}{\left(\frac{1}{1.15}\right)\left(1-\left(\frac{1}{1.15}\right)^{3}\right)} = 1313.93$$

So the annual repayments for each of the loans are £813.73, £1,769.84 and £1,313.93 respectively, and the total annual repayment is £3,897.50.

To find the amount outstanding on the loans, either find (i) the present value (now, not at the start of the loan period) of the difference between the loan amount and the payments already made, or (ii) find the present value of the future payments.

For loan 1, the amount outstanding using method (i) in pounds is

$$(1+i)^3(5000 - 813.73a_{\overline{3}|}) = 5000(1.1)^3 - 813.73(1.1)^3\frac{\left(\frac{1}{1.1}\right)\left(1-\left(\frac{1}{1.1}\right)^{3}\right)}{\left(1-\frac{1}{1.1}\right)} = 3961.55$$

For loan 1 using method (ii) the amount outstanding in pounds is

$$813.73a_{\overline{7}|} = 813.73\frac{\left(\frac{1}{1.1}\right)\left(1-\left(\frac{1}{1.1}\right)^{7}\right)}{\left(1-\frac{1}{1.1}\right)} = 3961.58$$

For loan 2, since only 1 payment has been made and $a_{\overline{1}|} = v$, the amount outstanding using method (i) in pounds is

$$(1+i)(10000 - 1769.84a_{\overline{1}|}) = 10000(1+i) - 1769.84 = 10000(1.12) - 1769.84 = 9430.16$$

For loan 2 using method (ii) the amount outstanding in pounds is

$$1769.84a_{\overline{9}|} = 1769.84\frac{\left(\frac{1}{1.12}\right)\left(1-\left(\frac{1}{1.12}\right)^{9}\right)}{\left(1-\frac{1}{1.12}\right)} = 9430.15$$

For loan 3, since only 2 payments have been made so $a_{\overline{2}|} = v + v^2$, the amount outstanding using method (i) in pounds is

$$(1+i)^2(3000 - 1313.93a_{\overline{2}|}) = 3000(1+i)^2 - 1313.93\left((1.15)+1\right) = 1142.55$$

For loan 3 using method (ii), since only 1 payment is still to be made, the amount outstanding in pounds is

$$1313.93a_{\overline{1}|} = 1313.93\frac{1}{1.15} = 1142.55$$

Results using methods (i) and (ii) differ marginally because of the rounding of the payments P_i.

The total amount outstanding using method (i) is $\pounds 3,961.55 + \pounds 9,430.16 + \pounds 1,142.55 = \pounds 14,534.26$

The total amount outstanding using method (ii) is $\pounds 3,961.58 + \pounds 9,430.15 + \pounds 1,142.55 = \pounds 14,534.28$

(b) I will take the amount outstanding to be £14,534.26 (and so you will obtain marginally different results if you used method (ii) in part (a)).

(i) We want £P, where $Pa_{\overline{10|}}$ is equal to the total amount outstanding. So

$$P = 14534.26 \frac{\left(1 - \frac{1}{1.06}\right)}{\left(\frac{1}{1.06}\right)\left(1 - \left(\frac{1}{1.06}\right)^{10}\right)} = 1974.74$$

So the repayment level for the secured loan is £1,974.74

(ii) The original total annual payment is $\pounds 813.73 + \pounds 1,769.84 + \pounds 1,313.93 = \pounds 3,897.50$ The schedule of payments is:

Year	Payment	Interest paid	Principal paid	Amount outstanding
0	0	0	0	14,534.26
1	3897.50	872.06	3,025.44	11,508.82
2	3,897.50	690.53	3,206.97	8,301.85
3	3,897.50	498.11	3,399.39	4902.46
4	3,897.50	294.15	3,603.35	1,299.11
5	1,377.06	77.95	1,299.11	0

Note that in year 5 the annual payment of $\pounds 3897.50$ was more than the interest and principal remaining. Hence the final repayment needed to clear the bank loan was only $\pounds 1,377.06$.