MAS224, Actuarial Mathematics: Solutions to Problem Sheet 2.

1. Interest is compounded monthly, hence $p = 12$.

   We wish to find the rate of discount per month, $\frac{d^{(12)}}{12}$, which is the amount to be paid in advance for use of 1 unit of money over one month, given that we know that the monthly rate of interest $\frac{i^{(12)}}{12}$ is 0.01. This is the amount of money to be paid in arrears for use of 1 unit of money over one month.

   Thus, $\frac{d^{(12)}}{12}$ is the present value of $\frac{i^{(12)}}{12}$ due in one month's time. Hence, $\frac{d^{(12)}}{12} = v^{1/12} \times \frac{i^{(12)}}{12}$.

   The discounting factor $v = \frac{1}{(1+i)}$ in terms of the monthly interest rate is given by $v = \frac{1}{(1+\frac{i^{(12)}}{12})^{12}}$, hence
   $$\frac{d^{(12)}}{12} = \frac{\frac{i^{(12)}}{12}}{1 + \frac{i^{(12)}}{12}} = \frac{0.01}{1 + 0.01} = 0.00990099$$

   So the rate of discount per month is 0.9901%.

   The amount of interest paid in arrears is $£1000\frac{i^{(12)}}{12} = £10$.

   The amount of interest paid in advance is $£1000\frac{d^{(12)}}{12} = £9.90$

   [Alternatively, the amount of interest paid in advance may be calculated as present value of $£10$, i.e. $£10v^{1/12} \equiv £\frac{10}{1+\frac{i^{(12)}}{12}} = £\frac{10}{1.01} = £9.90$]

2. The present value of the deferred annuity is $£10000v^2\ddot{a}_{\overline{10}|}$ and $v = \frac{1}{1+i} = \frac{1}{1+0.1} = \frac{10}{11}$. Hence the present value in pounds is just
   $$10000\,{}_{2}\ddot{a}_{\overline{10}|} = 10000v^2\ddot{a}_{\overline{10}|} = 10000\frac{v^2(1-v^{10})}{(1-v)} = \frac{10000\left(\frac{10}{11}\right)^2\left(1-\left(\frac{10}{11}\right)^{10}\right)}{\frac{1}{11}} = 55859.70096$$

   So the present value is $£55,859.70$

3. Here $i = 0.8$ so that the discounting factor $v = 1/1.08$.

   The loan is being repaid by an annuity-due with present value payable yearly at rate P per year over 25 years. The present value of this annuity is $P\ddot{a}_{\overline{25}|}$.

   The present value of the payments should equal the present value of the loan. Hence
   $$P\ddot{a}_{\overline{25}|} = 50000, \qquad \ddot{a}_{\overline{25}|} = \frac{1-v^{25}}{1-v}.$$

   Then
   $$P = \frac{50000(1-v)}{(1-v^{25})} = \frac{50000 \times \left(1 - \frac{1}{1.08}\right)}{1 - \left(\frac{1}{1.08}\right)^{25}} = 4336.980512$$

   The annual payment required is $£4,336.98$

4. Here $i = 0.1$ and so $v = \frac{1}{1+i} = 10/11$.

   The present value of the debts in pounds is $1000v + 1000v^2 + 5000v^4$.

   If the annual payment is P then the total present value of the annual payments is $Pa_{\overline{10}|}$. Note that we have an immediate annuity payable annually, at rate P per year, for 10 years.

To find P equate the present value of the debts to the present value of the payments:

$$Pa_{\overline{10}|} = 1000v + 1000v^2 + 5000v^4, \qquad a_{\overline{10}|} = \frac{v(1-v^{10})}{1-v},$$

and

$$P = \frac{(1000v + 1000v^2 + 5000v^4) \times (1-v)}{v(1-v^{10})} = 838.2371579.$$

Hence the annual payment required is £838.24.

If he is discharges the debts by a single payment of £$7,000$ at time t, equate the present value $7000v^t$ of this payment to the present value of the debts, then solve for t. So

$$7000v^t = 1000v + 1000v^2 + 5000v^4, \qquad v = \frac{10}{11}.$$

From this

$$t = \frac{\ln\left(\frac{1000v+1000v^2+5000v^4}{7000}\right)}{\ln v} = 3.2189.$$

Hence payment should be made in t years where $t = 3.2189$. This makes 3 years and 80 days.

5. Here $i = 0.04$ and so $v = \frac{1}{1.04}$.

   Have an annuity-due payable monthly at the rate of £$1,200$ per year for 10 years. The present value of this annuity is $1200\ddot{a}^{(12)}_{\overline{10}|}$,

$$1200\ddot{a}^{(12)}_{\overline{10}|} = 1200\frac{1-v^{10}}{12\left(1-v^{1/12}\right)} = 1200\frac{1-\left(\frac{1}{1.04}\right)^{10}}{12\left(1-\left(\frac{1}{1.04}\right)^{1/12}\right)} = 9942.694639.$$

   So the total present value of monthly payments is £9942.70.

   The present value of the lump sum is £$10,000$ which is more than the present value of the alternative prize, so take the prize in the form of lump sum.

6 Here $i = 0.08$ and so $v = \frac{1}{1.08}$.

   Part (i): Have a deferred annuity-due of £$13,000$ per year payable in equal monthly installments in perpetuity. Its present value in pounds is $13000 \times {}_{0.5}\ddot{a}^{(12)}_{\overline{\infty}|} = 13000 \times v^{1/2} \times \ddot{a}^{(12)}_{\overline{\infty}|}$ where

$$\ddot{a}^{(12)}_{\overline{\infty}|} = \lim_{n\to\infty} \ddot{a}^{(12)}_{\overline{n}|} = \lim_{n\to\infty} \frac{1}{12}\frac{1-v^n}{1-v^{1/12}} = \frac{1}{12}\frac{1}{1-v^{1/12}}.$$

   Thus

$$13000_{0.5}\ddot{a}^{(12)}_{\overline{\infty}|} = 13000\frac{v^{1/2}}{12(1-v^{1/12})} = 13000\frac{\left(\frac{1}{1.08}\right)^{1/2}}{12\left(1-\left(\frac{1}{1.08}\right)^{1/12}\right)} = 163061.8827.$$

   So the present value is £163,061.88

   Part (ii):

   Have a deferred annuity-due of £$13,000$ per year payable continuously (e.g. as an approximation think of equal daily payments) in perpetuity. Its present value in pounds is $13000 \times {}_{0.5}\overline{a}_{\overline{\infty}|} = 13000 \times v^{1/2} \times \overline{a}_{\overline{\infty}|}$ where

$$\overline{a}_{\overline{\infty}|} = \lim_{n\to\infty}\left(\lim_{p\to\infty} \ddot{a}^{(p)}_{\overline{n}|}\right) = \lim_{n\to\infty}\left(\lim_{p\to\infty} \frac{1}{p}\frac{1-v^n}{1-v^{1/p}}\right) = \lim_{n\to\infty} \frac{1-v^n}{-\ln v} = \frac{1}{\ln(1+i)}.$$

Thus

$$13000 \times {}_{0.5}\overline{a}_{\overline{\infty|}} = 13000\frac{v^{1/2}}{\ln(1+i)} = 13000\frac{\left(\frac{1}{1.08}\right)^{1/2}}{ln(1.08)} = 162540.1066.$$

So the present value is £162,540.11.

7 First find the accumulation by 1st April 2011. This is just the total present value of monthly payments times $(1+i)^3$.

The monthly payments are in the form of an annuity-due payable monthly at the rate of $\pounds 1200$ per year for 3 years. Hence the accumulation by 1st April 2011 is nothing else as $\pounds 1200(1+i)^3\ddot{a}^{(12)}_{\overline{3|}}$ where $i = 0.06$.

This is then invested for 9 months at 5% interest (so is scaled by $(1.05)^{3/4}$ to give the amount drawn out on 1st January 2012). Hence amount withdrawn in pounds is

$$(1.05)^{3/4}1200(1.06)^3\frac{\left(1-\left(\frac{1}{1.06}\right)^3\right)}{12\left(1-\left(\frac{1}{1.06}\right)^{1/12}\right)} = \frac{100(1.05)^{3/4}((1.06)^3-1)}{\left(1-\left(\frac{1}{1.06}\right)^{1/12}\right)} = 4090.346237$$

So the amount he receives is £4,090.35