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MAS224, Actuarial Mathematics: Solutions to Problem Sheet 2.

M-method Penalize for rounding off errors only once  
A - answer

Q1/5

M1

1. Interest is compounded monthly, hence  $p = 12$ .

We wish to find the rate of discount per month,  $\frac{d^{(12)}}{12}$ , which is the amount to be paid in advance for use of 1 unit of money over one month, given that we know that the monthly rate of interest  $\frac{i^{(12)}}{12}$  is 0.01. This is the amount of money to be paid in arrears for use of 1 unit of money over one month.

Thus,  $\frac{d^{(12)}}{12}$  is the present value of  $\frac{i^{(12)}}{12}$  due in one month's time. Hence,  $\frac{d^{(12)}}{12} = v^{1/12} \times \frac{i^{(12)}}{12}$ .

The discounting factor  $v = \frac{1}{(1+i)}$  in terms of the monthly interest rate is given by  $v = \frac{1}{(1+\frac{i^{(12)}}{12})^{12}}$ , hence

M1

or equiv.

$$\frac{d^{(12)}}{12} = \frac{\frac{i^{(12)}}{12}}{1 + \frac{i^{(12)}}{12}} = \frac{0.01}{1 + 0.01} = 0.00990099$$

A1

So the rate of discount per month is 0.9901%.

A1

The amount of interest paid in arrears is  $\mathcal{L}1000 \frac{i^{(12)}}{12} = \mathcal{L}10$ .

A1

The amount of interest paid in advance is  $\mathcal{L}1000 \frac{d^{(12)}}{12} = \mathcal{L}9.90$

[Alternatively, the amount of interest paid in advance may be calculated as present value of  $\mathcal{L}10$ , i.e.  $\mathcal{L}10v^{1/12} \equiv \mathcal{L} \frac{10}{1+\frac{i^{(12)}}{12}} = \mathcal{L} \frac{10}{1.01} = \mathcal{L}9.90$ ]

Q2/3

M2

2. The present value of the deferred annuity is  $\mathcal{L}10000v^2\ddot{a}_{\overline{10}|}$  and  $v = \frac{1}{1+i} = \frac{1}{1+0.1} = \frac{10}{11}$ . Hence the present value in pounds is just

or equiv.

$$10000 {}_2\ddot{a}_{\overline{10}|} = 10000v^2\ddot{a}_{\overline{10}|} = 10000 \frac{v^2(1-v^{10})}{(1-v)} = \frac{10000 \left(\frac{10}{11}\right)^2 \left(1 - \left(\frac{10}{11}\right)^{10}\right)}{\frac{1}{11}} = 55859.70096$$

A1

Q3/3

3. Here  $i = 0.08$  so that the discounting factor  $v = 1/1.08$ .

The loan is being repaid by an annuity-due with present value payable yearly at rate  $P$  per year over 25 years. The present value of this annuity is  $P\ddot{a}_{\overline{25}|}$ .

The present value of the payments should equal the present value of the loan. Hence

M2

or equiv

$$P\ddot{a}_{\overline{25}|} = 50000, \quad \ddot{a}_{\overline{25}|} = \frac{1-v^{25}}{1-v}$$

Then

$$P = \frac{50000(1-v)}{(1-v^{25})} = \frac{50000 \times \left(1 - \frac{1}{1.08}\right)}{1 - \left(\frac{1}{1.08}\right)^{25}} = 4336.980512$$

A1

Q4/5

4. Here  $i = 0.1$  and so  $v = \frac{1}{1+i} = 10/11$ .

The present value of the debts in pounds is  $1000v + 1000v^2 + 5000v^4$ .

If the annual payment is  $P$  then the total present value of the annual payments is  $P\ddot{a}_{\overline{10}|}$ . Note that we have an immediate annuity payable annually, at rate  $P$  per year, for 10 years.

To find  $P$  equate the present value of the debts to the present value of the payments:

M2

or equiv.

$$Pa_{\overline{10}|} = 1000v + 1000v^2 + 5000v^4, \quad a_{\overline{10}|} = \frac{v(1-v^{10})}{1-v},$$

and

$$P = \frac{(1000v + 1000v^2 + 5000v^4) \times (1-v)}{v(1-v^{10})} = 838.2371579.$$

A1

Hence the annual payment required is £838.24.

If he discharges the debts by a single payment of £7,000 at time  $t$ , equate the present value  $7000v^t$  of this payment to the present value of the debts, then solve for  $t$ . So

M1

accept if using P.V. of debts from part one

$$7000v^t = 1000v + 1000v^2 + 5000v^4, \quad v = \frac{10}{11}.$$

From this

$$t = \frac{\ln\left(\frac{1000v + 1000v^2 + 5000v^4}{7000}\right)}{\ln v} = 3.2189.$$

A1

Hence payment should be made in  $t$  years where  $t = 3.2189$ . This makes 3 years and 80 days.

accept any of these two

Q5/4

5. Here  $i = 0.04$  and so  $v = \frac{1}{1.04}$ .

Have an annuity-due payable monthly at the rate of £1,200 per year for 10 years. The present value of this annuity is  $1200\ddot{a}_{\overline{10}|}^{(12)}$ ,

M2

$$1200\ddot{a}_{\overline{10}|}^{(12)} = 1200 \frac{1-v^{10}}{12(1-v^{1/12})} = 1200 \frac{1 - \left(\frac{1}{1.04}\right)^{10}}{12\left(1 - \left(\frac{1}{1.04}\right)^{1/12}\right)} = 9942.694639.$$

A1

So the total present value of monthly payments is £9942.70.

A1

if say 10,000 is more than P.V. of annuity.

The present value of the lump sum is £10,000 which is more than the present value of the alternative prize, so take the prize in the form of lump sum.

6 Here  $i = 0.08$  and so  $v = \frac{1}{1.08}$ .

Part (i): Have a deferred annuity-due of £13,000 per year payable in equal monthly installments in perpetuity. Its present value in pounds is  $13000 \times 0.5\ddot{a}_{\infty|}^{(12)} = 13000 \times v^{1/2} \times \ddot{a}_{\infty|}^{(12)}$  where

$$\ddot{a}_{\infty|}^{(12)} = \lim_{n \rightarrow \infty} \ddot{a}_{\overline{n}|}^{(12)} = \lim_{n \rightarrow \infty} \frac{1}{12} \frac{1-v^n}{1-v^{1/12}} = \frac{1}{12} \frac{1}{1-v^{1/12}}.$$

Thus

$$13000 \times 0.5\ddot{a}_{\infty|}^{(12)} = 13000 \frac{v^{1/2}}{12(1-v^{1/12})} = 13000 \frac{\left(\frac{1}{1.08}\right)^{1/2}}{12\left(1 - \left(\frac{1}{1.08}\right)^{1/12}\right)} = 163061.8827.$$

So the present value is £163,061.88

Part (ii):

Have a deferred annuity-due of £13,000 per year payable continuously (e.g. as an approximation think of equal daily payments) in perpetuity. Its present value in pounds is  $13000 \times 0.5\bar{a}_{\infty|} = 13000 \times v^{1/2} \times \bar{a}_{\infty|}$  where

$$\bar{a}_{\infty|} = \lim_{n \rightarrow \infty} \left( \lim_{p \rightarrow \infty} \ddot{a}_{\overline{n}|}^{(p)} \right) = \lim_{n \rightarrow \infty} \left( \lim_{p \rightarrow \infty} \frac{1}{p} \frac{1-v^n}{1-v^{1/p}} \right) = \lim_{n \rightarrow \infty} \frac{1-v^n}{-\ln v} = \frac{1}{\ln(1+i)}.$$

no marks for Q6

Thus

$$13000 \times 0.5 \bar{a}_{\infty|} = 13000 \frac{v^{1/2}}{\ln(1+i)} = 13000 \frac{\left(\frac{1}{1.08}\right)^{1/2}}{\ln(1.08)} = 162540.1066.$$

So the present value is £162,540.11.

Q7 | 5

7 First find the accumulation by 1st April 2011. This is just the total present value of monthly payments times  $(1+i)^3$ .

M3  $\leftarrow$  1 for 1200  
1 for  $\times (1+i)^3$   
1 for  $\bar{a}_{\frac{12}{3}|}$

The monthly payments are in the form of an annuity-due payable monthly at the rate of £1200 per year for 3 years. Hence the accumulation by 1st April 2011 is nothing else as  $\boxed{\text{£}1200(1+i)^3 \bar{a}_{\frac{12}{3}|}^{(12)}}$  where  $i = 0.06$ . or equiv.

M1 Scale by  $(1.05)^{3/4}$

This is then invested for 9 months at 5% interest (so is scaled by  $(1.05)^{3/4}$  to give the amount drawn out on 1st January 2012). Hence amount withdrawn in pounds is

$$(1.05)^{3/4} 1200 (1.06)^3 \frac{\left(1 - \left(\frac{1}{1.06}\right)^3\right)}{12 \left(1 - \left(\frac{1}{1.06}\right)^{1/12}\right)} = \frac{100(1.05)^{3/4} ((1.06)^3 - 1)}{\left(1 - \left(\frac{1}{1.06}\right)^{1/12}\right)} = 4090.346237$$

A1

So the amount he receives is £4,090.35