Total 25 marks

A is for answer, M is for method.

MAS224, Actuarial Mathematics: Solutions to Problem Sheet 1.

Q1 3 marks

1. The nominal rate per annum, $i^{(12)}$, when interest is compounded monthly is 12 times the interest rate per month. Here the rate per month is 0.005, so $i^{(12)} = 12 \times 0.005 = 0.06$. Giving the interest rate as a percentage as required, the nominal interest rate per annum for interest compounded monthly is $\boxed{6\%.}$

A1

To find the corresponding annual effective rate, we use the relationship $1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p$. Writing the annual effective rate of interest, i, in terms of $i^{(12)}$,

M1

$$\boxed{i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1} = (1 + 0.005)^{12} - 1 = 0.061677811$$

So (giving the answer as a percentage to 4dp as required), the annual effective rate of interest is $\boxed{6.1678\%.}$

A1

Q2 4 marks

2. Here $i = 0.15$ and $p = 52$. We want $i^{(52)}$. Write $i^{(52)}$ in terms of i.

M1

$$\boxed{i^{(52)} = 52\left((1+i)^{1/52} - 1\right)} = 52\left((1+0.15)^{1/52} - 1\right) = 0.139949931$$

A1

So the nominal rate of interest per annum when interest is compounded weekly is $\boxed{13.9950\%.}$

The accumulation of an investment of £5,000 over a period of 13 weeks can be written either in terms of $i^{(52)}$ or in terms of i. Either form can be used for the calculation, though it is easiest to use the exact value of i given in the question. If you use $i^{(52)}$ you should use more decimal places than in the rounded answer above.

M1

$$\text{Accumulation} = \boxed{£5000\left(1 + \frac{i^{(52)}}{52}\right)^{13}} \equiv £5000(1+i)^{1/4} = £5000(1.15)^{0.25} = £5177.79038$$

A1

So the accumulation (to the nearest penny as required) is $\boxed{£5,177.79}$

Q3 6 marks

M2 (one mark if at least one formula correct)

A1

3. Here $\delta = 0.2$ and you need to write $i, i^{(4)}, i^{(12)}$ and $i^{(365)}$ in terms of δ.

(a)

$$i = \left(e^{\delta} - 1\right) = \left(e^{0.2} - 1\right) = 0.221402758$$

So the annual interest rate is $\boxed{22.1403\%.}$

(b)

$$i^{(4)} = 4\left(e^{\delta/4} - 1\right) = 4\left(e^{0.2/4} - 1\right) = 0.205084385$$

A1

So the nominal interest rate per annum when interest is compounded quarterly is $\boxed{20.5084\%.}$

(c)

$$i^{(12)} = 12\left(e^{\delta/12} - 1\right) = 12\left(e^{0.2/12} - 1\right) = 0.201675964$$

A1

So the nominal interest rate per annum when interest is compounded monthly is $\boxed{20.1676\%.}$

(d)

$$i^{(365)} = 365\left(e^{\delta/365} - 1\right) = 365\left(e^{0.2/365} - 1\right) = 0.200055$$

A1.

So the nominal interest rate per annum when interest is compounded daily is $\boxed{20.0055\%.}$

Q4 3marks

M2

4. Let the annual effective rate of interest be i. Then the accumulation of £10,000 over five years is just $\boxed{£10,000(1+i)^5}$. This has to be equal to £20,000. Hence $\boxed{(1+i)^5 = 2}$, and so

$$i = 2^{1/5} - 1 = 0.148698355$$

A1

So the APR is $\boxed{14.8698\%}$.

Q5 6marks

5. Bank A. Here $i^{(52)} = 0.11$ and the amount payable is the principal and interest after a period of two years (i.e. 104 weeks). So the amount payable is

M1

$$\boxed{£10000\left(1+\frac{i^{(52)}}{52}\right)^{104}} = £10000\left(1+\frac{0.11}{52}\right)^{104} = £12457.87$$

A1

If the loan is taken out with bank A then the amount payable is $\boxed{£12,457.87}$

Bank B. Here $i^{(4)}/4 = 0.03$ and the amount payable is the principal and interest after a period of two years (i.e. 8 quarters). So the amount payable is

M1

$$\boxed{£10000\left(1+\frac{i^{(4)}}{4}\right)^{8}} = £10000\,(1+0.03)^8 = £12667.70$$

A1

If the loan is taken out with bank B then the amount payable is $\boxed{£12,667.70}$

Bank C. For the first six months, $i = 0.03$. So at the end of 6 months the principal plus interest on the loan is $£10000(1+0.03)^{0.5}$. Over the next one and a half years the annual interest rate $i = 0.15$. By the end of this period the amount payable is

M1

$$\boxed{£10000(1+0.03)^{0.5}(1+0.15)^{1.5}} = £12515.99$$

A1

If the loan is taken out with bank C then the amount payable is $\boxed{£12,515.99}$

Clearly you would prefer to pay out less money, so that you would choose bank A for your loan.

Q6 3marks

M1

6. Accumulated amount is $\boxed{£1000e^{\int_0^4 \delta(t)dt}}$. Now, splitting the range, $\int_0^4 \delta(t)dt$ is equal to

M(Integral) 1

$$\boxed{\int_0^2 0.05dt + \int_2^4 0.05(t-1)dt} = [0.05t]_{t=0}^{t=2} + [0.025(t-1)^2]_{t=2}^{t=4} = 0.1 + 0.025(9-1) = 0.3$$

A1

So the accumulated amount is $£1000e^{0.3}$, i.e. $\boxed{£1,349.86}$

7. (a) We have to show that $i^{(p+1)} < i^{(p)}$ for all positive p.

As $i^{(p)} = p(e^{\frac{\delta}{p}} - 1)$, it will suffice to show that the derivative of $p(e^{\frac{\delta}{p}} - 1)$ with respect to p is negative.

$$(p(e^{\frac{\delta}{p}} - 1))' = e^{\frac{\delta}{p}}\left(1 - \frac{\delta}{p}\right) - 1$$

Now by making use of the inequality $e^{-x} > 1-x$ with $x = \delta/p > 0$, or equivalently $e^{x}(1-x) < 1$, we conclude that

$$(p(e^{\frac{\delta}{p}} - 1))' < 0.$$

(b) Have to show that $\lim_{p\to\infty} i^{(p)} = \delta$.

Rewrite $i^{(p)} = p(e^{\frac{\delta}{p}} - 1)$ in the form

$$i^{(p)} = \frac{(e^{\frac{\delta}{p}} - 1)}{\frac{1}{p}}.$$

When you take the limit as p tends to infinity, the numerator and denominator in the ratio above tend to zero. So you use l'Hopital's Rule, which says that the limit of the ratio is the limit of the ratio of the derivatives. Therefore

$$\begin{aligned}
\lim_{p\to\infty} i^{(p)} &= \lim_{p\to\infty} \frac{(e^{\delta/p}-1)}{1/p} \\
&= \lim_{p\to\infty} \frac{(-\delta/p^2)e^{\delta/p}}{-1/p^2} \\
&= \lim_{p\to\infty} \delta e^{\delta/p} = \delta
\end{aligned}$$