MAS224, Actuarial Mathematics: Problem Sheet 9

This problem sheet should not be submitted; it is only intended for you to have examples to do from Section 4. The solutions will made made available on the web at the end of term.

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1. Beetles in a population die before they reach age 3. The birth rate and survival distribution depend only on the age of the female and not on her year of birth. Females in the population reproduce during their first and second years only (i.e. at ages 0 and 1), each producing on average 1 new living females at age 0 and 4 new living females at age 1. The probability that a beetle survives its first year of life is 1/2 and the probability that a beetle survives its second year of life is 1/3. Let $n_x(t)$ be the expected number aged x in the population at time t and let $\mathbf{n}(t)$ be the vector of the $n_x(t)$ for $x = 0, 1, 2$.

   Write down the Leslie matrix M for this population. Find the values of the survival function $S(x)$ for $x = 0, 1, 2$. Find the expected number of female offspring produced during the lifetime of a single female beetle. Is there a solution with $n_x(t) = n_x$ for $x = 0, 1, 2$ and all non-negative integers t?

   Find the Euler solution.

   Suppose that there are nine hundred females in each age group at time $t = 0$. Use the equation for $\mathbf{n}(t)$ to find the expected numbers of females in each age group at times $t = 1$, $t = 2$ and $t = 3$.

2. Females in a population die before they reach age 3. The birth rate and survival distribution depend only on the age of the female and not on her year of birth. They only reproduce at age 1, each producing on average 2 new female offspring. The probability that a female survives her first year of life is 1/2 and the probability that she then survives her second year of life is 1/4.

   Write down the Leslie matrix M for this population.

   (i) Show that $\sum_{x=0}^{2} b_x S(x) = 1$. Find the eigenvalues of $\mathbf{M}$ and show that the only real positive eigenvalue is equal to 1. Obtain the Euler solution with $\lambda = 1$.

   (ii) Show that $M^3 = M$. If $n_0(0) = n_1(0) = n_2(0) = 100$, find $n_i(t)$ for $t = 1, 2$ and $i = 0, 1, 2$. Hence find $n_i(t)$ for all integers $t > 2$ and all $i = 0, 1, 2$.

3. Consider a female who is age x at time t. Let $b_x(t)$ be the expected number of female offspring she has at that age and let $p_x(t)$ be the probability she survives to age $x + 1$. Denote the expected number of females age x in the population in year t by $n_x(t)$. Then $n_x(t+1) = p_x(t)n_x(t)$, for $x = 0, 1, \ldots$, and $n_0(t+1) = \sum_{x=0}^{\infty} b_x(t)n_x(t)$. Then Euler's model with $p_x(t) \equiv p_x$, $b_x(t) \equiv b_x$, $n_x(t) = \lambda^t n_x$ satisfies these equations, provided that λ satisfies a characteristic equation which has a unique real positive root.

   State this equation and find the root when the distribution of the lifetime is exponential, so that the survival function $S(x) = e^{-\theta x}$, in each of the following cases: (i) $b_x = e^{-\alpha x}$ for $x = 0, 1, \ldots$; (ii) $b_0 = 0$ and $b_x = \alpha$ for $x = 1, 2, \ldots$. In each case find the condition on α and θ which ensures that the population is growing.