LECTURE 9

Basic life-table functions; Force of mortality

Notations

X is the age-at-death (lifetime) of a newborn in a population.

Assume: X is a continuous-type random variable and $F_X(0) = 0$.

Survival function: $s(x) = P(X > x) = 1 - F_X(x)$

$s(x)$ is the probability that a newborn will attain age x.

Distribution of X can be specified either by $F_X(x)$ or by $s(x)$: $f_X(x) = \frac{d}{dx}F_X(x) = -\frac{d}{dx}s(x)$

The symbol (x) is used to denote a *life-age-x*. The future lifetime of (x) is denoted by $T(x)$. $T(x)$ is the time-until-death for a person age x. Using conditioning by the event $X > x$, the p.d.f of $T(x)$ can be expressed in terms of the survival function:

$$f_{T(x)}(t) \equiv f_{(X-x)|(X>x)}(t) = \frac{f_X(x+t)}{s(x)} \quad \text{[by (k) from Lecture 7]} \tag{1}$$

$$= -\frac{1}{s(x)}\frac{d}{dx}s(x+t). \tag{2}$$

The symbol ${}_tq_x$ is used to denote the probability that (x) will die within t years:

$$\begin{aligned} {}_tq_x &= P(T(x) \le t) \\ &= P(x < X \le x+t \mid X > x) \\ &= \frac{P(x < X \le x+t)}{P(X > x)} \quad \text{[by (a) and (e) from Lecture 7]} \\ &= \frac{F_X(x+t) - F_X(x)}{1 - F_X(x)} \quad \text{[by (i) from Lecture 7]} \\ &= \frac{s(x) - s(x+t)}{s(x)} \quad \text{using } s(x) = 1 - F_X(x) \\ &= 1 - \frac{s(x+t)}{s(x)} \end{aligned}$$

The symbol ${}_tp_x$ is used to denote the probability that (x) will attain age $x+t$:

$$\begin{aligned} {}_tp_x = 1 - {}_tq_x &= P(T(x) > t) \\ &= P(X > x+t \mid X > x) \\ &= \frac{P(X > x+t)}{P(X > x)} \quad \text{[by (a) and (e) from Lecture 7]} \\ &= \frac{s(x+t)}{s(x)} \end{aligned} \tag{3}$$

When $t = 1$ the prefix t is omitted:

$$\begin{aligned} p_x &= P((x) \text{ will attain age } x+1) = P(X > x+1 \mid X > x) \\ q_x &= P((x) \text{ will die within 1 year}) = P(x < X \le x+1 \mid X > x) \end{aligned}$$

The symbol ${}_{t|u}q_x$ is used to denote the probability that (x) will survive t years and die within the following u years, i.e.

$$\begin{aligned} {}_{t|u}q_x &= P(t < T(x) \le t+u) \\ &= P(x+t < X \le x+t+u \mid X > x) \\ &= \frac{s(x+t) - s(x+t+u)}{s(x)} \end{aligned}$$

The following relations can be verified directly:

$$\begin{aligned} {}_{t|u}q_x &= {}_tp_x - {}_{t+u}p_x \\ &= \left({}_tp_x\right) \times \left({}_uq_{x+t}\right) \end{aligned}$$

Force of Mortality

(conventional notation for which is μ)

This is defined as

$$\mu(x) = -\frac{1}{s(x)}\frac{d}{dx}s(x) = -\frac{d}{dx}\ln s(x). \tag{4}$$

Note: that for small Δx

$$\begin{aligned} \mu(x)\Delta x &\doteq \frac{s(x) - s(x+\Delta x)}{s(x)} \\ &= P(x < X \le x+\Delta x \mid X > x) \end{aligned}$$

Hence $\mu(x)\Delta x$ is the conditional probability that a newborn will die in the age interval $(x, x+\Delta x]$ given survival to age x.

The force of mortality is often interpreted as the instantaneous death rate or instantaneous rate of mortality.

Assume that we have a population of l_0 individuals whose age-at-death is described by a survival function which is identical for all individuals. The number of individuals who die aged x is a random variable. Its expectation value $q_x \times l_0$. Thus

q_x can be interpreted as the (expected) rate of mortality at age x.

Similarly, the expected number of individuals who die between ages x and $x+u$ is $l_0 \times ({}_u q_x)$. Hence $\frac{1}{u} \times ({}_u q_x)$ is the rate of deaths p.a. in the age interval $(x, x+u]$.

$$\begin{aligned} \frac{1}{u} \times ({}_u q_x) &= \frac{P(x < X \le x+u \mid X > x)}{u} \\ &= \frac{1}{s(x)} \left[\frac{s(x) - s(x+u)}{u} \right] \end{aligned}$$

and, in the limit $u \to 0$, $\frac{1}{u} \times ({}_u q_x) \to \mu(x)$. Thus $\mu(x)$ is the "instantaneous rate of mortality ", i.e. the rate of *instant* deaths at age x.

As is true for the survival function, the force of mortality can be used to specify the distribution of X (age-at-death). Indeed, by our assumption, $F_X(0) = 0$, hence $s(0) = 1$ and by integrating Eq. (4):

$$s(x) = e^{-\int_0^x \mu(u)du}.$$

In addition,

$$F_X(x) = 1 - s(x) = 1 - e^{-\int_0^x \mu(u)du}.$$

From Eqs. (3):

$${}_t p_x = e^{-\int_0^t \mu(x+u)du} = e^{-\int_x^{x+t} \mu(u)du}.$$

It follows from Eqs. (1)-(2) and (4) that the p.d.f. for $T(x)$ can be expressed in terms of force of mortality:

$$\begin{aligned} f_{T(x)}(t) &= -\frac{1}{s(x)} \frac{d}{dt} s(x+t) = -\frac{s(x+t)}{s(x)} \frac{1}{s(x+t)} \frac{d}{d(x+t)} s(x+t) \\ &= {}_t p_x \mu(x+t). \end{aligned}$$

Thus, for small Δt, ${}_t p_x \mu(x+t)\Delta t$ is the probability that (x) dies between ages $x+t$ and $x+t+\Delta t$.

SUMMARY.

Survival function $s(x) = P(X > x)$:	$s(x) = 1 - F_X(x)$
Force of mortality:	$\mu(x) = -\frac{d}{dx} \ln s(x)$
$T(x)$ is the future lifetime of (x)	p.d.f.: $f_{T(x)}(t) = {}_t p_x \mu(x+t)$
	${}_t p_x = P(T(x) > t) = \frac{s(x+t)}{s(x)}$
	${}_t q_x = P(T(x) \le t) = \frac{s(x) - s(x+t)}{s(x)}$
	${}_{t\|u} q_x = P(t < T(x) \le t+u) = \frac{s(x+t) - s(x+t+u)}{s(x)}$