LECTURE 8

Life annuities: an introduction

In practice an annuity is paid only whilst the individual receiving it is alive. The human lifetime is uncertain. It depends on many factors, such as the state of health, genes inherited from parents, habits, employment, weather conditions, area, and many others. Hence the number of payments to be made with an annuity is uncertain too. One has to take into account this factor when estimating the cost of annuity.

In the context of actuarial investigations, the uncertainty of human life is modelled by a survival model where the lifetime of an individual is a random variable. Cost and present values are estimated by the expectation value of the quantity of interest over the lifetime distribution.

Example: Exponentially distributed lifetime (Toy Model).

Assume that the future lifetime for the newborns, X, has exponential distribution with parameter λ.
Suppose that John on his x-th birthday is taking an immediate annuity of 1 p.a. for life. We want to find an estimate for the cost of John's annuity.
Note: As $X \sim \text{Exp}(\lambda)$, $E(X) = \frac{1}{\lambda}$, hence

$$\begin{array}{ll} \lambda \to 0, & \text{"live forever";} \\ \lambda \to \infty, & \text{"die instantly".} \end{array}$$

The cost of John's annuity is nothing else as its present value. It depends on $X - x$, the future lifetime for John who is age x now. This future lifetime is uncertain, as we do not know exactly X. However, we know the probability law for X. Hence, we can as an estimate for the cost its mean value.
The probabilities that are relevant for evaluating this mean value are conditional. We cannot use the probabilities for X directly. For example, $P(t < X - x \leq s)$ is the probability that a newborn dies between ages $x+t$ and $x+s$. This takes into account the possibility that this newborn will not survive to age x or age $x+t$. Regarding John, we know that he has survived to age x, so the probability for him to die between ages $x+t$ and $x+s$ is not the same as for a newborn!
The probability that John dies between ages $x+t$ and $x+s$ can be expressed through probabilities for the newborns using conditioning by the event $X > x$:

$$P(\text{John dies between ages } x+t \text{ and } x+s) = P(t < X - x \leq s | X > x)$$

The annuity is paid at the end of every year that John survives. The present value (P.V.) of John's annuity is a random variable:

$$\text{P. V.} = \begin{cases} 0, & \text{if} \quad 0 < X - x \leq 1, \\ a_{\overline{1}|}, & \text{if} \quad 1 < X - x \leq 2, \\ a_{\overline{2}|}, & \text{if} \quad 2 < X - x \leq 3, \\ ... & ... \end{cases}$$

where X is John's age at death.

Therefore,

$$\text{P. V.} = \begin{cases} 0, & \text{with probability} \quad P(0 < X - x \leq 1 \mid X > x) \\ a_{\overline{1}|}, & \text{with probability} \quad P(1 < X - x \leq 2 \mid X > x) \\ a_{\overline{2}|}, & \text{with probability} \quad P(2 < X - x \leq 3 \mid X > x) \\ ... & ... \end{cases}$$

Using the usual rule for computing the mean value of random variables, the expected present value (E.P.V.) of Johns annuity is

$$\text{E.P.V.} = \sum_{j=1}^{\infty} a_{\overline{j}|}\, P(j < X - x \leq j+1 \mid X > x).$$

As X is exponentially distributed, the two random variables X and $(X-x)|(X > x)$ have the same distribution. Hence,

$$P(j < X - x \leq j+1 \mid X > x) = P(j < X \leq j+1)$$

and

$$\begin{aligned} \text{E.P.V.} &= \sum_{j=1}^{\infty} a_{\overline{j}|}\, P(j < X \leq j+1) = \sum_{j=1}^{\infty} a_{\overline{j}|}\, [F_X(j+1) - F_X(j)] \\ &= \sum_{j=1}^{\infty} \frac{v(1-v^j)}{1-v} [e^{-\lambda j} - e^{-\lambda(j+1)}] = \frac{v}{1-v} \sum_{j=1}^{\infty} (1-v^j)(e^{-\lambda j} - e^{-\lambda} e^{-\lambda j}), \end{aligned}$$

as $F_X(t) = 1 - e^{\lambda t}$ and $a_{\overline{j}|} = \frac{v(1-v^j)}{1-v}$, $v = \frac{1}{1+i} = e^{-\delta}$, δ being the force of interest.

Using

$$\sum_{j=1}^{\infty} q^j = \frac{q}{1-q} \quad \text{with } q = ve^{-\lambda},$$

we finally obtain that

$$\text{E.P.V.} = \frac{1}{(1+i)e^{\lambda} - 1} = \frac{1}{e^{(\delta+\lambda)} - 1}.$$

Note that:

in the limit $\lambda \to 0$ (live forever) E.P.V. $\to \frac{1}{i} \equiv a_{\overline{\infty}|}$ and
in the limit $\lambda \to \infty$ (die instantly) E.P.V. $\to 0$.