LECTURE 21

Whole-life annuity: Expectation and variance of the present value

P.V. of life annuities is a random variable. It can be exprssed through the indicator-of-survival function $\mathbb{1}_k$. E.g. the P/V. of a whole-life annuity-due of 1 p.a. is $z = \sum_{k=0}^{\infty} v^k \mathbb{1}$ (Eq. (1) in Lecture 20). This representation is very convenient for the purpose of evaluating the expectation of z but is less useful in calculations of the variance of z. The reason is that the random variables $\mathbb{1}_k$, $k = 0, 1, \ldots$, are not independent and hence $\text{var}(z) = \text{var}(\sum_k v^k \mathbb{1}_k) \neq \sum_k \text{var}(v^k \mathbb{1}_k)$.

It is often more convenient to express the present values of life annuities directly in terms of exact time-until-death, $T(x)$, or its integer part, $K(x)$, known as the curtate time-until-death (see Lecture 10).

Recall from Lecture 5 that the present value of a continuous flow of money over time interval $[0, t]$ at the rate 1 p.a. is

$$\bar{a}_{\overline{t|}} = \int_0^t v^u \, du = \frac{1 - v^t}{\delta}, \qquad \delta = -\ln v = \ln(1 + i) \tag{1}$$

If (x) is entitled to 1 p.a. for life payable continuously, the corresponding payment is a whole-life annuity. On the other hand, this payment is to be made continuously over the time interval $[0, T(x)]$, hence its present value is $\bar{a}_{\overline{T(x)|}}$.

Therefore:

> the present value of a whole-life annuity payable continuously, at rate 1 p.a., until the moment of death of (x) is $\bar{a}_{\overline{T(x)|}}$, hence $\bar{a}_x = E(\bar{a}_{\overline{T(x)|}})$.

The present value of an annuity-due of 1 p.a. payable annually (at the beginning of each year) is

$$\ddot{a}_{\overline{k+1|}} = 1 + v + \ldots v^k = \frac{1 - v^{k+1}}{1 - v} = \frac{1 - v^{k+1}}{d}, \tag{2}$$

where $k + 1$ is the number of payments and d is the effective rate of discount.

Therefore, similarly to annuities payable continuously:

> $\ddot{a}_{\overline{K(x)+1|}}$ is the present value of a whole-life annuity of 1 p.a. payable annually in advance until the death of (x). As the annuity is payable in advance, the first installment is due at the present time (i.e. now) and is payable even if $K(x) = 0$, hence $K(x) + 1$, not $K(x)$. Obviously, $\ddot{a}_x = E(\ddot{a}_{\overline{K(x)+1|}})$.

Expected present value of annuities, revisited

Let us first re-derive Eq. (2) of Lecture 20 which expresses the present value $\bar{a}_x$ through the survival probabilities ${}_tp_x$ and v. Now we shall start with $\bar{a}_x = E(a_{\overline{T(x)}|})$:

$$\begin{aligned}\bar{a}_x = E(\bar{a}_{\overline{T(x)}|}) &= \int_0^\infty \bar{a}_{\overline{t}|}\, f_{T(x)}(t)\,dt = \int_0^\infty \bar{a}_{\overline{t}|}\; {}_tp_x \mu(x+t)dt \\ &= -\int_0^\infty \bar{a}_{\overline{t}|}\, d({}_tp_x) = -\bar{a}_{\overline{t}|}\, {}_tp_x \Big|_0^\infty + \int_0^\infty {}_tp_x \Big(\frac{d}{dt}\bar{a}_{\overline{t}|}\Big) dt.\end{aligned}$$

Notice that $\bar{a}_{\overline{t}|}\, {}_tp_x \Big|_0^\infty = 0$ and $\frac{d}{dt}\bar{a}_{\overline{t}|} = v^t$, the latter follows from Eq. (1). Therefore

$$\bar{a}_x = E(\bar{a}_{\overline{T(x)}|}) = \int_0^\infty v^t\, {}_tp_x\, dt,$$

the same expression for $\bar{a}_x$ as in Eq. (2) in Lecture 20.

Now whole-life annuities-due.

$\ddot{a}_{\overline{K(x)+1}|}$ is a discrete-type random variable that takes the values $\ddot{a}_{\overline{k+1}|}$ with the probabilities $P(K(x) = k) = {}_kp_x - {}_{k+1}p_x$ (see Lecture 10). Therefore

$$\begin{aligned}\ddot{a}_x = E(\ddot{a}_{\overline{K(x)+1}|}) &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} P(K(x) = k) \\ &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} ({}_kp_x - {}_{k+1}p_x).\end{aligned}$$

Now notice that $\ddot{a}_{\overline{k+1}|} = \ddot{a}_{\overline{k}|} + v^k$ for all integer k (follows from Eq. (2)). Therefore

$$\begin{aligned}E(\ddot{a}_{\overline{K(x)+1}|}) &= \sum_{k=0}^{\infty} (\ddot{a}_{\overline{k}|} + v^k)({}_kp_x) - \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}({}_{k+1}p_x) \\ &= 1 + \sum_{k=1}^{\infty} v^k\, {}_kp_x \qquad [\text{as } \ddot{a}_{\overline{0}|} = 0]\end{aligned}$$

Thus finally,

$$\ddot{a}_x = E(\ddot{a}_{\overline{K(x)+1}|}) = \sum_{k=0}^{\infty} v^k\, {}_kp_x,$$

the same expression for $\ddot{a}_x$ as that obtained in Lecture 20!

Variance of the present value of annuities

We want to express var($\bar{a}_{\overline{T(x)}|}$), variance of the present value of a whole-life annuity payable continuously, and var($\ddot{a}_{\overline{K(x)+1}|}$), variance of the present value of a whole-life annuity payable annually in advance, in terms of the life table functions.

From Probability I, for any random variable X and two constants α and β: $\mathrm{var}(\alpha X+\beta)=\alpha^2\mathrm{var}(X)$. We will use this fact and Eqs. (1) and (2) to find $\mathrm{var}(\bar{a}_{\overline{T(x)|}})$ and $\mathrm{var}(\ddot{a}_{\overline{K(x)+1|}})$

By Eq. (1),

$$\mathrm{var}(\bar{a}_{\overline{T(x)|}})=\frac{1}{\delta^2}\mathrm{var}(v^{T(x)}).$$

From Lecture 16, $v^{T(x)}$ is the present value of one unit of benefit payable immediately on death under a whole-life assurance contract and its variance is $\bar{A}_x^*-(\bar{A}_x)^2$, where $\bar{A}_x=E(v^{T(x)})$ and the star refers to $v^*=v^2$. Therefore

$$\mathrm{var}(\bar{a}_{\overline{T(x)|}})=\frac{\bar{A}_x^*-(\bar{A}_x)^2}{\delta^2}. \tag{3}$$

Similarly, by Eq. (2),

$$\mathrm{var}(\ddot{a}_{\overline{K(x)+1|}})=\frac{1}{d^2}\mathrm{var}(v^{K(x)+1}),$$

where $v^{K(x)+1}$ is the present value of one unit of benefit payable at the end of the year of death under a whole-life assurance contract (see Lecture 18) and $\mathrm{var}(v^{K(x)+1})=A_x^*-(A_x)^2$. Therefore,

$$\mathrm{var}(\ddot{a}_{\overline{K(x)+1|}})=\frac{A_x^*-(A_x)^2}{d^2}. \tag{4}$$

Conversion Relationships:

$$\bar{A}_x = 1-\delta\bar{a}_x \tag{5}$$

$$A_x = 1-d\,\ddot{a}_x \tag{6}$$

Eqs. (5) and (6) follow from Eqs. (1) and (2). Indeed, from Eq. (1), $v^{T(x)}=1-\delta\bar{a}_{\overline{T(x)|}}$ and from Eq. (2), $v^{K(x)+1}=1-d\,\ddot{a}_{\overline{K(x)+1|}}$. Taking the expectation one obtains Eqs. (5) and (6).