Force of interest
(conventional notation for which is δ)

If interest is applied, at a particular time t, to a fund of value $F(t)$, the value of the fund will grow in time dt to $F(t+dt)$. The force of interest is then defined as

$$\delta(t) = \lim_{h\downarrow 0} \frac{F(t+h)-F(t)}{F(t)\times h}. \tag{1}$$

As $F(t+h)-F(t) \simeq \delta(t)\times[F(t)\times h]$ for small h, the force of interest is the *instantaneous rate of growth per unit capital and per unit time*.

Assume that $F(t)$ and $\delta(t)$ are both continuous functions of time. Then one can rewrite (1) as

$$\delta(t) = \frac{1}{F(t)}\frac{d}{dt}F(t) = \frac{d}{dt}\ln F(t).$$

By integrating this, $\int_0^t \delta(s)ds = \ln F(t) - \ln F(0)$ and

$$F(t) = F(0)\exp\left[\int_0^t \delta(s)ds\right]. \tag{2}$$

Notation:
$A(t_1,t_2) =$ the accumulation at time t_2 of one unit of money invested at time t_1.

Assumptions:

(i) Principle of Consistency: $A(t_1,t_3) = A(t_1,t_2)A(t_2,t_3)$ for any $t_1 < t_2 < t_3$, i.e. the accumulation does not depend on when and how often money is withdrawn and reinvested.

(ii) The rate of interest does not vary with the size of investment.

In practice, it is rarely that (i) and (ii) are realized exactly. For instance, savings accounts offer a better rate of interest for a larger investment. However, the rate of interest does not vary within intervals of sizes of investment.

Notice that

$$A(t_1,t_2) = \frac{A(0,t_2)}{A(0,t_1)} = \frac{F(t_2)/F(0)}{F(t_1)/F(0)} = \exp\left[\int_{t_1}^{t_2}\delta(s)ds\right] \tag{3}$$

This is a very important result.

Nominal rates of interest
Time unit:- usually one year, but need not be specified for general theory.

By the principle of consistency, the accumulation of one unit investment over n unit periods of time is

$$\begin{aligned} A(0,n) &= A(0,1)\times A(1,2)\times\ldots\times A(n-1,n) \\ &= [1+i(0)]\times[1+i(1)]\times\ldots\times[1+i(n-1)], \end{aligned}$$

where $i(s)$ is the rate of interest for one unit time period beginning at time s.

For accumulations over time periods other than one unit of time:- nominal rates are in use.

Nominal rate of interest is a rate, per unit time, of interest which applies over a *different* time period. Example: overnight money – yearly rate of interest which is applied daily (i.e. interest is converted into capital daily, but interest is quoted as a rate per annum).

Define $i^{(p)}(t)$ as the nominal rate, at time t, of interest convertible p-thly , i.e. the rate of interest *per unit time* applied to time period $[t, t+\frac{1}{p}]$. As $i^{(p)}(t)$ expresses the interest per unit time, we have

$$A\left(t, t+\frac{1}{p}\right) = 1+\frac{i^{(p)}(t)}{p}. \tag{4}$$

Obviously, $i^{(1)}(t)$ is just $i(t)$, the annual rate of interest compounded annually.

Example.

$i^{(4)}(0)$ is the annual rate applied from $t=0$ to $t=\frac{1}{4}$, $i^{(2)}(\frac{1}{4})$ is the annual rate applied from $t=\frac{1}{4}$ to $t=\frac{1}{4}+\frac{1}{2}=3/4$, $i^{(4)}(\frac{3}{4})$ is the annual rate applied from $t=\frac{3}{4}$ to $t=\frac{3}{4}+\frac{1}{4}=1$. Thus, by the Principle of Consistency

$$1+i^{(1)}(0) = \left(1+\frac{i^{(4)}(0)}{4}\right)\left(1+\frac{i^{(2)}(1/4)}{2}\right)\left(1+\frac{i^{(4)}(3/4)}{4}\right).$$

Another interpretation of δ:

$$\delta(t) = \lim_{p\to\infty} i^{(p)}(t), \tag{5}$$

i.e. δ is the nominal rate of interest *convertible instantly*.

To prove (5) notice that definition (1) can be written in terms of A:

$$\delta(t) = \lim_{h\downarrow 0}\frac{A(t,t+h)-1}{h}.$$

Hence

$$\begin{aligned} \delta(t) &= \lim_{p\to\infty}\frac{A\left(t,t+\frac{1}{p}\right)-1}{\frac{1}{p}} \\ &= \lim_{p\to\infty} i^{(p)}(t) \qquad \text{[by (4)]} \end{aligned}$$

Although δ is a theoretical concept, all Compound Interest formulae can be derived from (3).

A very important practical case:-

$$\delta(t) = \delta = \text{constant over all } t.$$

In this case for all t and all n

$$A(t,t+n) = e^{\delta n} \qquad \text{[follows from (3)].} \qquad (6)$$

Here n may be fractional, e.g. $A(t,t+\frac{1}{p}) = e^{\frac{\delta}{p}}$ when $n = 1/p$. From this

$$\begin{aligned} i^{(p)} &= p(e^{\frac{\delta}{p}} - 1) && \text{[by (4)]} && (7) \\ i &= e^{\delta} - 1; && \text{[as } i^{(1)} = i\text{]} \\ e^{\delta} &= 1 + i. && && (8) \end{aligned}$$

From Eqs. (6) and (8), $A(t,t+n) = (1+i)^n$. This is just Eq(2) from Lecture 1 but we have shown it to be true for all values $n \geq 0$, not just integers.

In the rest of this course a constant force of interest will be assumed, unless explicitly stated otherwise.

When δ does not vary with time the nominal rates stay constant and apply over each of the time periods $[\frac{k}{p}, \frac{k+1}{p}]$, $k = 0,1,\ldots$, hence over any unit time period. The annual rate of interest i is also constant over time and applies over any unit time period. The basic relationship between i and $i^{(p)}$ is

$$1 + i = \left[1 + \frac{i^{(p)}}{p}\right]^p \qquad \text{[this follows from (6) and (8)]} \qquad (9)$$

i is often called the *effective annual rate* of interest to distinguish it from nominal rates.

Annual Percentage Rate of Charge (APR).

This is defined as the effective annual rate of interest on a transaction, obtained by taking into account all the items entering the total charge for credit. When only interest is charged on a loan, the APR on the loan is just the effective annual rate of interest. By the Consumer Credit Act (1974), the total charge for credit and APR have to be disclosed in advertisements and in quotations for consumer credit agreements.

SUMMARY.

Accumulation at time t_2 of a unit investment at time t_1:	$A(t_1,t_2) = \exp\left[\int_{t_1}^{t_2} \delta(s)ds\right]$
When interest is converted p-thly:	$A\left(t,t+\frac{1}{p}\right) = 1 + \frac{i^{(p)}(t)}{p}$
Force of interest:	$\delta(t) = \lim_{h\downarrow 0} \frac{A(t,t+h)-1}{h} = \lim_{p\to\infty} i^{(p)}(t)$
Constant force of interest δ:	$e^{\delta} = 1 + i = \left[1 + \frac{i^{(p)}}{p}\right]^p$

Worked Examples

Worked Example 1. (Nominal rates)

On 2 October 1998 the nominal rates of interest per annum for local authority deposits in London were (source: Financial Times on 3/10/98):

Term (period of notice)	1 day	7 days	One month	3 mths	Six mths	One year
Nominal rate of interest (%)	7	$7\frac{1}{4}$	$7\frac{11}{32}$	$7\frac{5}{16}$	$7\frac{3}{16}$	$6\frac{7}{8}$

Find the accumulation of an investment at this time of £10,000 for (a) one day (overnight money) and (b) one week

Solution.

(a) $p = 365$; $i^{(365)} = 0.07$.
By Eq. (4), the accumulation is £10000 $\times\left[1 + \frac{0.07}{365}\right]$=£10,001.92

(b) $p = \frac{365}{7}$; $i^{(\frac{365}{7})} = 0.0725$.
By Eq. (4), the accumulation is £10000 $\times\left[1 + \frac{7\times 0.0725}{365}\right]$=£10,013.90

Worked Example 2. (Nominal rates)

The force of interest per unit time, δ, where time is measured in years, equals 0.12. Find the nominal rate of interest per annum on deposits of term (a) one day, (b) seven days, (c) one month, (d) 3 months, and (e) 6 months.

Solution.

Use $i^{(p)} = p[e^{\frac{\delta}{p}} - 1]$, this is Eq. (7).

(a) $\delta = 0.12$, $p = 365$; $\quad i^{(365)} = 0.12002$ or 12.002% .
(b) $\delta = 0.12$, $p = \frac{365}{7}$; $\quad i^{(\frac{365}{7})} = 0.12014$ or 12.014% .
(c) $\delta = 0.12$, $p = 12$; $\quad i^{(12)} = 0.12060$ or 12.06% .
(d) $\delta = 0.12$, $p = 4$; $\quad i^{(4)} = 0.12182$ or 12.182% .
(e) $\delta = 0.12$, $p = 2$; $\quad i^{(2)} = 0.12367$ or 12.367% .

Worked Example 3. (Monthly repayments)

The nominal rate of interest compounded monthly is 12%. Find the corresponding effective annual rate (same as APR).

Solution.

$i^{(12)} = 0.12$. By the basic relationship between $i^{(12)}$ and the corresponding effective annual rate, Eq. (9), $i = \left[1 + \frac{i^{(12)}}{12}\right]^{12} - 1 = 0.1268$, or 12.68%

Worked Example 4. (Force of interest)

Assume that the force of interest varies with time and is given by $\delta(t) = a + \frac{b}{t}$. Find the formula for the accumulation of one unit money from time t_1 to time t_2.

Solution.

By Eq. (3),

$$A(t_1,t_2) = \exp\left[\int_{t_1}^{t_2}(a+\frac{b}{s})ds\right] = \exp\left[a(t_2-t_1)\right]\left(\frac{t_2}{t_1}\right)^b$$

Approximate expressions for δ and i in terms of $i^{(p)}$ when p is large

We know $\delta = \ln(1+i) = p\ln[1+\frac{i^{(p)}}{p}]$; [follows from Eqs. (8) and (9)]

Use Taylor's expansion of $\ln(1+x)$, $|x|<1$ (Calculus I) to obtain:-

$$\begin{aligned} \delta &= i^{(p)} - \frac{[i^{(p)}]^2}{2p} + \varepsilon, && \text{where } |\varepsilon| \le \frac{[i^{(p)}]^3}{3p^2} \qquad (10) \\ &= i^{(p)} - \frac{[i^{(p)}]^2}{2p} + \frac{[i^{(p)}]^3}{3p^2} + \varepsilon, && \text{where } |\varepsilon| \le \frac{[i^{(p)}]^4}{4p^3} \\ &\dots \end{aligned}$$

Compare these relations with relation (5).

For example, if we know $i^{(12)}$, the nominal rate of interest compounded monthly, then

$$\begin{aligned} \delta &= i^{(12)} - \frac{\left[i^{(12)}\right]^2}{24} + \varepsilon \\ &\simeq i^{(12)} - \frac{\left[i^{(12)}\right]^2}{24} \qquad (11) \end{aligned}$$

If $0 \le i^{(52)} \le 0.5$ then $|\varepsilon| < 0.0003$, so the approximate relation (11) is in fact quite accurate.

Formula (10) can be used to give another proof of the relation $\delta = \lim_{p\to\infty} i^{(p)}$ (the force if interest is the nominal rate of interest compounded instantly). Try to find this proof.