LECTURE 19

Life assurances, continued

Present value of death benefit payments due at the end of the year of death

Recall from lecture 10:

$K(x)$ = the number of complete years lived by (x) = integer part of $T(x)$.

$K(x)$ is a discrete-type random variable with p.m.f.

$$P(K(x) = k) = P(k \le T(x) < k+1) = P(T(x) > k) - P(T(x) > k+1) = {}_kp_x - {}_{k+1}p_x.$$

Whole life assurance

Death benefit: 1 unit of money payable at the end of the year of death of (x). Its P.V. in terms of $K(x)$, is

$$z = v^{K(x)+1}. \tag{1}$$

The expectation of z (i.e. E.P.V. of 1 unit of money due at the end of the year of death of (x)) is denoted by the symbol A_x.

$$\begin{aligned}
A_x = E(z) &= E(v^{K(x)+1}) \\
&= \sum_{k=0}^{\infty} v^{k+1} P(K(x) = k) \\
&= \sum_{k=0}^{\infty} v^{k+1} ({}_kp_x - {}_{k+1}p_x) \\
&= \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+k} - l_{x+k+1}}{l_x} \\
&= \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+k}}{l_x} - \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+k+1}}{l_x} \\
&= \sum_{k=0}^{\infty} \frac{vD_{x+k}}{D_x} - \sum_{k=0}^{\infty} \frac{D_{x+k+1}}{D_x} \\
&= \sum_{k=0}^{\infty} \frac{vD_{x+k}}{D_x} - \sum_{k=1}^{\infty} \frac{D_{x+k}}{D_x} \\
&= \sum_{k=0}^{\infty} \frac{vD_{x+k}}{D_x} - \left(\sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} - 1 \right) \\
&= \frac{vN_x}{D_x} - \frac{N_x}{D_x} + 1 \\
&= 1 - (1-v)\frac{N_x}{D_x}.
\end{aligned}$$

Thus

$$A_x = 1 - d\,\frac{N_x}{D_x}, \tag{2}$$

where d is the rate of discount, $d = 1 - v = \frac{i}{1+i}$.

As $E(z^2) = E\left((v^2)^{K(x)+1}\right)$, we have

$$\begin{aligned} \operatorname{var}(z) &= E(z^2) - [E(z)]^2 \\ &= E\left((v^2)^{K(x)+1}\right) - [E(v^{K(x)+1})]^2 \\ &= A_x^* - (A_x)^2, \end{aligned}$$

where A_x^* refers to the interest rate i^* such that $v^* = v^2$ and can be evaluated using Eq. (2) with d replaced by $d^* = 1 - v^* = 1 - \frac{1}{1+i^*}$, where $i^* = i^2 + 2i$.

n-year term life assurance

Under this type of policy the death benefit is paid only if the life assured (x) dies within the term of the policy. That is, if the life assured survives to age $x+n$ no benefit payment is made. Hence, to express the corresponding present value in terms of $K(x)$ we have to modify $z = v^{K(x)+1}$ (Eq. (1)) cutting it off at age $x+n$ (equivalently at $K(x) = n$):

$$z = \begin{cases} v^{K(x)+1}, & \text{if } K(x) < n \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

This is the P.V. of one unit of benefit under the n-year term life assurance with death benefit payable at the end of the year of death.

The expectation of z (E.P.V.) is denoted by the symbol $A^1_{x:\overline{n|}}$ and

$$\begin{aligned} A^1_{x:\overline{n|}} &= E(z) \\ &= \sum_{k=0}^{n-1} v^{k+1} P(K(x) = k) \\ &= \sum_{k=0}^{n-1} v^{k+1} ({}_k p_x - {}_{k+1}p_x) \\ &= \sum_{k=0}^{n-1} v^{k+1} \frac{l_{x+k} - l_{x+k+1}}{l_x} \\ &= \sum_{k=0}^{n-1} \frac{vD_{x+k}}{D_x} - \sum_{k=0}^{n-1} \frac{D_{x+k+1}}{D_x} \\ &= \sum_{k=0}^{n-1} \frac{vD_{x+k}}{D_x} - \sum_{k=1}^{n} \frac{D_{x+k}}{D_x} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{vD_{x+k}}{D_x} - \left(\sum_{k=0}^{n-1} \frac{D_{x+k}}{D_x} + \frac{D_{x+n}}{D_x} - 1 \right) \\
&= 1 - d \sum_{k=0}^{n-1} \frac{D_{x+k}}{D_x} - \frac{D_{x+n}}{D_x} \\
&= 1 - d \, \frac{N_x - N_{x+n}}{D_x} - \frac{D_{x+n}}{D_x}.
\end{aligned}
$$

Therefore,

$$A^1_{x:\overline{n}|} = 1 - d \, \frac{N_x - N_{x+n}}{D_x} - \frac{D_{x+n}}{D_x}.$$

The corresponding variance is

$$A^1_{x:\overline{n}|@i^*} - [A^1_{x:\overline{n}|}]^2$$

in line with prevoiusly obtained expressions.

Notice that, as $A_{x:\overline{n}|}^{\;\;1} = \frac{D_{x+n}}{D_x}$, we can also write

$$A^1_{x:\overline{n}|} = 1 - d \frac{N_x - N_{x+n}}{D_x} - A_{x:\overline{n}|}^{\;\;1}, \tag{4}$$

where $A_{x:\overline{n}|}^{\;\;1} = \frac{D_{x+n}}{D_x}$ is the E.P.V. of 1 due on the survival of (x) to age $x+n$ (pure endowment policy).

n-year term endowment policy

The present value of benefits (both equal 1 unit of money and the death benefit is payable at the end of the year of death) is

$$z = \begin{cases} v^{K(x)+1}, & \text{if} \quad K(x) < n \\ v^n, & \text{if} \quad K(x) \geq n \end{cases}$$

Its expected present value is denoted by the symbol $A_{x:\overline{n}|}$ and

$$
\begin{aligned}
A_{x:\overline{n}|} = E(z) &= E(z_1) + E(z_2) \\
&= A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1},
\end{aligned}
$$

where z_1 is the P.V. of Eq. (3) and z_2 is the P.V. of Eq. (2) of Lecture 18 (recall that the endowment policy is a combination of a temporary life assurance and a pure endowment).

From Eq. (4) we obtain an expression for $A_{x:\overline{n}|}$ in terms of the commutation functions N_x and D_x:

$$A_{x:\overline{n}|} = 1 - d \, \frac{N_x - N_{x+n}}{D_x}$$