LECTURE 15

Life Table as a Population Model

Life Table as a Model of a Stationary Population

A population, from the demographic point of view, is described by the relative numbers of individuals in the age groups. The time evolution of the age distribution in a population is affected by two factors: mortality and reproduction. We will consider this question in more detail later on in the course when dealing with population models.

Here I just briefly explain how a life table can be used to describe a stationary population.

Denote by $n_x(t)$ the expected number of individuals aged x in the population at time t. The set of functions $n_x(t)$, $x = 0, 1, 2, \ldots$ describes the age decomposition in the population.

We will assume that

(i) mortality rates are described by a survival function which does not change over time, i.e. $P(X > x) = s(x)$ at any time t.

(ii) the age decomposition do not change over time, i.e. $n_x(t) = n_x$ for all t.

It is obvious that

$$n_{x+1} = n_x p_x \qquad \text{for all } x = 0, 1, 2, \ldots \tag{1}$$

Let us count deaths in one year in the population. The expected number of all deaths when added up over all age grous is $\sum_{x=0}^{infty} n_x q_x$.

$$\begin{aligned}
\sum_{x=0}^{\infty} n_x q_x &= \sum_{x=0}^{\infty} n_x (1 - p_x) \\
&= \sum_{x=0}^{\infty} n_x - \sum_{x=0}^{\infty} n_x p_x \\
&= \sum_{x=0}^{\infty} n_x - \sum_{x=0}^{\infty} n_{x+1} \\
&= n_0,
\end{aligned}$$

that is in our population the expected number of deaths in one your equals n_0, the expected number of births in one year.

Aside: Showing that $s(x) = p_{x-1} p_{x-2} \cdot \ldots \cdot p_0$.

Recall that $P(A\cap B) = P(A|B)P(B)$. Put $A = (X > x)$ and $B = (X > x-1)$ in this relation. Obviously $(X > x)\cap(X > x-1) = (X > x)$ and therefore,

$$P(X > x) = P(X > x|X > x-1)P(X > x-1).$$

But $P(X > x) = s(x)$, $P(X > x-1) = s(x-1)$ and $P(X > x|X > x-1) = p_{x-1}$. Therefore

$$\begin{aligned} s(x) &= s(x-1)p_{x-1} = s(x-2)p_{x-1}p_{x-2} = \dots \\ &= s(0)p_{x-1}p_{x-2}\cdot\ldots\cdot p_0 \\ &= p_{x-1}p_{x-2}\cdot\ldots\cdot p_0 \end{aligned}$$

as $s(0) = 1$.

From (1),

$$\begin{aligned} n_x &= n_{x-1}p_{x-1} = n_{x-2}p_{x-1}p_{x-2} = \dots \\ &= n_0 p_{x-1}p_{x-2}\cdot\ldots\cdot p_0 \end{aligned}$$

and

$$n_x = n_0 s(x)$$

as $s(x) = p_{x-1}p_{x-2}\cdot\ldots\cdot p_0$.

The age structure of population can be modelled by an appropriate life table. Indeed, if the life table' mortality rates agree with those of the population then

$$n_x = n_0 s(x) \quad \text{and} \quad l_x = l_s(0).$$

$$\therefore \qquad \frac{l_x}{l_0} = \frac{n_x}{n_0} \quad \text{and} \quad n_x = \frac{n_0}{l_0} l_x.$$

The factor $A = \frac{n_0}{l_0}$ is called the scaling factor.

Calculating the expected population size and numbers of aged x and over

The expected population size N:

$$\begin{aligned} N &= \sum_{k=0}^{\infty} n_k = A\sum_{k=0}^{\infty} l_k = A\left(l_0 + \sum_{k=1}^{\infty} l_k\right) = A(l_0 + l_0 e_0) = \frac{n_0}{l_0} l_0 (1 + e_0) \\ &\doteq n_0\left(\frac{1}{2} + \mathring{e}_0\right) \qquad \text{as } e_x \doteq \mathring{e}_x - \frac{1}{2} \text{ for all } x. \end{aligned}$$

Similarly, the expected number of individuals aged x and over, $N(x)$:

$$
\begin{aligned}
N(x) &= \sum_{k=x}^{\infty} n_x = A\sum_{k=x}^{\infty} l_k = A\left(l_x + \sum_{k=x+1}^{\infty} l_k\right) = A(l_x + l_x \mathrm{e}_x) = \frac{n_0}{l_0} l_x\,(1+\mathrm{e}_x) \\
&\simeq \frac{n_0}{l_0} l_x \left(\frac{1}{2} + \mathring{\mathrm{e}}_x\right).
\end{aligned}
$$

Worked Example 1. Population as a Life Table

Suppose that we have a stationary population of 10,000,000. The mortality rates are represented by the English Life Table No. 12 – Males. Find (a) the expected number of births per year, and (b) the expected number of pensioners

Solution.

(a) $N = n_0(\mathring{\mathrm{e}}_0 + {}^1\!/_2)$. $N = 10,000,000$ and $\mathring{\mathrm{e}}_0 = 68.09$ from the table. Hence,

$$
n_0 = \frac{N}{\mathring{\mathrm{e}}_0 + {}^1\!/_2} = \frac{10000000}{68.59} = 145,794.
$$

(b) $N(65) = \frac{n_0}{l_0} l_{65}(\mathring{\mathrm{e}}_{65} + {}^1\!/_2) = \frac{145794}{100000} \times 68490 \times (11.95 + 0.5) = 1,243,185.$