

LECTURES 14-15

*Interpolation of life-table values for fractional ages**Linear Interpolation:*(has been used in the derivation of $\ddot{e}_x = e_x + \frac{1}{2}$)

If we know value of $s(x)$ only for integer x we can approximate $s(y)$ in the interval between x and $x+1$ by the linear function:

$$s(x+t) \simeq (1-t)s(x) + ts(x+1), \quad x=0, 1, 2, \dots, 0 < t < 1.$$

This approximation implies that

$$\begin{aligned} {}_{k+t}p_x &\simeq {}_kp_x(1-t) + {}_{k+1}p_x t & x, k=0, 1, 2, \dots, 0 < t < 1. \\ l_{x+t} &\simeq (1-t)l_x + tl_{x+1}, & x=0, 1, 2, \dots, 0 < t < 1, \\ &= l_x - td_x. \end{aligned}$$

The expected number of deaths in the interval $[x+t, x+t+\Delta t]$ is

$$\Delta t d_{x+t} = l_{x+t} - l_{x+t+\Delta t}.$$

Applying linear interpolation when Δt is small,

$$\Delta t d_{x+t} \simeq l_x - td_x - [l_x - (t + \Delta t)d_x] = \Delta t d_x.$$

This means that the density of deaths within the age interval between x and $x+1$ is d_x , i.e. constant. Therefore linear interpolation implies uniform distribution of deaths within each year of age. The inverse statement is also true: uniform distribution of deaths within each year of age implies piecewise linear survival function.

Therefore, linear interpolation is consistent with the assumption of uniform distribution of deaths within each year of age.

Also in use in actuarial science:-

Exponential interpolation

$$\ln s(x+t) \simeq (1-t)\ln s(x) + t\ln s(x+1), \quad x=0, 1, 2, \dots \text{ and } 0 < t < 1.$$

Exponential interpolation is consistent with the assumption of a constant force of mortality within each year of age.

Harmonic interpolation (also known as hyperbolic or Balducci assumption)

$$\frac{1}{s(x+t)} \simeq (1-t)\frac{1}{s(x)} + t\frac{1}{s(x+1)}, \quad x=0, 1, 2, \dots \text{ and } 0 < t < 1.$$

Force of mortality cannot be observed. Its values (even those for integer x given in life tables) has to be calculated from the values of other life-table functions. Linear interpolation is not suitable for this purpose as it gives two different values of $\mu(x)$, one when it is used over the interval $[x-1, x]$ and the other when it is used over $[x, x+1]$. Because of this other approximations are usually used. Four most common are:

(1) Based on $p_x = e^{-\int_0^1 \mu(x+t)dt}$.

$$\int_0^1 \mu(x+t)dt = -\ln p_x \quad \therefore \mu\left(x + \frac{1}{2}\right) \simeq -\ln p_x$$

(2) Based on ${}_2p_{x-1} = e^{-\int_{-1}^1 \mu(x+t)dt}$ (follows from ${}_2p_{x-1} = p_{x-1}p_x$).

$$-\ln(p_{x-1}p_x) = \int_{-1}^1 \mu(x+t)dt \simeq 2\mu(x) \quad \therefore \mu(x) \simeq -\frac{1}{2}(\ln p_x + \ln p_{x-1})$$

(3) Based on the assumption that l_y is a quadratic polynomial in y in the interval $[x-1, x+1]$.

$$\mu(x) \simeq \frac{l_{x-1} - l_{x+1}}{l_x}.$$

(4) Based on the assumption that l_y is a quartic polynomial in y in the interval $[x-2, x+2]$.

$$\mu(x) \simeq \frac{8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})}{12l_x}.$$

Example.

Using approximation (4) and the values of l_x for ages 38, 39, 40, 41, 42 from the English Life Table No. 12 – Males, estimate $\mu(40)$

$$\begin{aligned} \mu(x) &\simeq \frac{8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})}{12l_x} \\ &= \frac{8(93991 - 93570) - (94176 - 93328)}{12 \times 93790} = 0.00224. \end{aligned}$$

The obtained value coincides with the value of μ_{40} in the table.