MAS224, Actuarial Mathematics: Test solutions 2007-8

Present values of annuities-certain:

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{1-v} \qquad a_{\overline{n}|} = v\ddot{a}_{\overline{n}|}$$

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{1}{p}\left[\frac{1-v^n}{1-v^{\frac{1}{p}}}\right] \qquad a^{(p)}_{\overline{n}|} = v^{\frac{1}{p}}\ddot{a}^{(p)}_{\overline{n}|}.$$

1. **(5 marks)** If the rate of interest is 0.75% *per month* when interest is compounded monthly, find the nominal rate of interest and the annual equivalent rate (AER). Give your answers as percentages and to 2 decimal places.

◇ $i^{(12)} = 12 \times 0.0075 = 0.09.$ Nominal rate 9% per year.

◇ Use $1+i = \left(1+\frac{i^{(12)}}{12}\right)^{12}$.

$i = \left(1+\frac{i^{(12)}}{12}\right)^{12} - 1 = (1+1.0075)^{12} - 1 = 0.0938$ (4 d.p.)

AER=9.38%

How much interest should be paid in arrears for the use of £1,000 for one year? Give your answer to the nearest penny.

◇ $1000 \times 0.0938 = 93.80$ (2 d.p.).

Interest to be paid in arrears £93.80

How much interest should be paid in advance for the use of £1,000 for one year? Give your answer to the nearest penny.

◇ This is just the discounted value of the interest to be paid in advance.

EITHER

$93.80/(1+i) = 93.80/1.0938 = 85.76$ (2 d.p.).

OR

$1000 \times i/(1+i) = 1000 \times 0.0938/1.0938 = 85.76$ (2 d.p.).

Interest to be paid in advance £85.76

2. (4 marks) A loan of £10,000 is to be repaid by payments of £P per month in arrears for 10 years. Find the monthly payment if the annual interest rate is 7%. Give your answer to the nearest penny.

$\diamond$ Monthly payments in arrears - use $a^{(12)}_{\overline{10}|}$.

If P per month, then $12P$ per year, scale by $12P$.

Equation of value $10000 = 12P \times a^{(12)}_{\overline{10}|}$

$P = \frac{10000}{12 \times a^{(12)}_{\overline{10}|}} = \frac{10000(1-v^{1/12})}{v^{1/12}(1-v^{10})} = \frac{10000(1-(1/1.07)^{1/12})}{(1/1.07)^{1/12}(1-(1/1.07)^{10})} = 115.00$ (2 d.p.)

Monthly repayment of £115.00

3. (3 marks) Suppose that £1,000 is to be invested annually in advance for 5 years in an account paying interest at 5% per annum. Find the total accumulated value of this investment just after the final payment has been made. Give your answer to the nearest penny.

$\diamond$ Find P.V. of investment and then scale by $(1+i)^5$ to obtain accumulated value.

Have annual payments in advance for 5 years, use $\ddot{a}_{\overline{5}|}$.

£1000 per year, multiply $\ddot{a}_{\overline{5}|}$ by 1000.

P.V. (in pounds) is $1000 \times \ddot{a}_{\overline{5}|}$.

$1000\ddot{a}_{\overline{5}|} \times (1+i)^5 = 1000 \times \frac{1-(1/1.05)^5}{1-(1/1.05)} \times (1.05)^5 = 5801.91$ (2 d.p.)

Accumulated value £5,801.91

4. (3 marks) Joe Bloggs has a debt of £2,000 due 2 years from now and another one of £1,000 due 3 years from now. If Joe is allowed to discharge these debts by a single payment of £P in one year's time, what should this payment be if the interest is charged at 10% per annum? Give your answer to the nearest penny.

$\diamond$ Equation of value: $2000v^2 + 1000v^3 = Pv$

$P = 2000v + 1000v^2 = 2000 \times (1/1.1) + 1000 \times (1/1.1)^2 = 2644.63$ (2 d.p.)

Payment £2,644.63.

5. **(8 marks)** Suppose that beetles in a particular population are certain to die before they reach age 4 and their mortality is described by the survival function $s(x) = \frac{4-x}{4}$ for $0 \le x \le 4$.

Find the instantaneous death rate (force of mortality) at age 2. Use this result to obtain an approximation for the expected number of beetles out of 1,000 alive at age 2 who die within one week of reaching that age (i.e. age 2). Give your answer to the nearest integer.

◇ $\mu(2) = -\frac{s'(2)}{s(2)}$

$s(2) = \frac{1}{2}$ and $s'(x) = -\frac{1}{4}$ for all $0 \le x \le 4$.

$\mu(2) = 0.5$.

◇ The expected number of dying is approximately $\mu(2) \times 1000 \times \frac{1}{52}$.

$\frac{1}{2} \times 1000 \times \frac{1}{52} = 10$ (to the nearest integer)

Expect 10 beetles to dye.

If $K(x)$ is the curtate further lifetime at age x, find the probability mass function of $K(2)$. Hence find e_2, the curtate further expectation of life at age 2, for this population of beetles.

◇ $K(2)$ takes two values 0 and 1.

Use either $P(K(2) = k) = {}_{k|1}q_2$, or $P(K(2) = k) = {}_k p_2 - {}_{k+1}p_2$.

$P(K(2) = 0) = 1 - p_2 = 1 - \frac{s(3)}{s(2)} = 1 - \frac{1/4}{1/2} = \frac{1}{2}$,

$P(K(2) = 1) = {}_1p_2 - {}_2p_2 = {}_1p_2 = \frac{s(3)}{s(2)} = \frac{1/4}{1/2} = \frac{1}{2}$

$K(2)$ takes two values 0 and 1 with equal probabilities.

◇ EITHER

$\mathrm{e}_2 = E(K(2)) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$,

OR

$\mathrm{e}_2 = \frac{1}{s(2)}(s(3) + s(4)) = \frac{1}{\frac{1}{2}}\left(\frac{1}{4} + 0\right) = \frac{1}{2}$.

6. (12 marks) Use the life table ELT12 to obtain the following results:

(a) Calculate to 4 decimal places the probability that a life aged 60 survives to age 80.

◇ $_{20}p_{60} = \frac{l_{80}}{l_{60}} = \frac{22933}{78924} = 0.2906$ (to 4 d.p.)

(b) Calculate the expected number of deaths between age 60 and 65 out of 1000 newborns. Give your answer to the nearest integer.

◇ EITHER scale the expected number of deaths, i.e. $\frac{1000}{l_0}{}_5d_{60} = \frac{1000(l_{60}-l_{65})}{100000} = 0.01(78924-68490) = 104$ (to the nearest integer).

OR use the result that the number of deaths has $Bin(1000, {}_{60|5}q_0)$ distribution so that the expected number of deaths is $1000_{60|5}q_0 = 1000\frac{l_{60}-l_{65}}{l_0} = 1000\frac{(78924-68490)}{100000} =$ 104 (to the nearest integer).

104 deaths to be expected.

(c) Express $_tq_x$ in terms of the survival function.

◇ $_tq_x = \frac{s(x)-s(x+t)}{s(x)}$

Let x be an integer and let $0 < t < 1$. Use linear interpolation of the survival function between integer ages to show that $_tq_x = t \times q_x$.

◇ $_tq_x = \frac{s(x)-s(x+t)}{s(x)} = \frac{s(x)-((1-t)s(x)+ts(x+1))}{s(x)} = \frac{t(s(x)-s(x+1))}{s(x)} = t \times q_x$

Find the probability that a life who has just reached his 65^{th} birthday dies within the next 9 months. Give your answer to 4 significant figures.

◇ This is just $_{(\frac{9}{12})}q_{65} = {}_{(\frac{3}{4})}q_{65}$ so you can use the result just obtained.

$_{(\frac{3}{4})}q_{65} = \frac{3}{4} \times q_{65} = \frac{3}{4} \times 0.03648 = 0.02736$ (to 4 s.f.)

END