Actuarial Mathematics Test Solutions 2007

1. Interest is compounded monthly. The monthly rate of interest is 1%,

   (i) Find the annual equivalent rate of interest.

   The nominal rate $i^{(12)}$ is 12 times the monthly rate. So $i^{(12)} = 0.12$. You are asked to find i.

   The relation you need is $(1+i) = \left(1+\frac{i^{(12)}}{12}\right)^{12}$, i.e. $i = \left(1+\frac{0.12}{12}\right)^{12} - 1 = 0.126825$. Hence the APR is 12.6825%

   (ii) Find the annual equivalent rate of discount.

   This is just d. You can either write d in terms of i (so $(1-d) = \frac{1}{(1+i)}$ i.e. $d = \frac{i}{1+i}$) and use the first answer or write d in terms of $i^{(12)}$ (so $(1-d) = \left(1+\frac{i^{(12)}}{12}\right)^{-12}$).

   This gives $d = \frac{i}{1+i} = \frac{0.126825}{1.126825} = 0.112551$, so the annual equivalent rate of discount is 11.2551%.

   (iii) How much interest should be paid in arrears for the use of £5,000 over a one year period?

   Thia is just $£5000i = £634.13$

   (iv) How much interest should be paid in advance for the use of £5,000 over a one year period?

   This is just $£5000d = £562.75$.

2. Regular savings of £100 per month are made in advance for 3 years. The APR paid on the account is 5% per annum. Find the accumulation at the end of the 3 year period.

   This is just $£1200\ddot{a}^{(12)}_{\overline{3}|}(1+i)^3$ where $i = 0.05$. So the accumulation in pounds is just

$$1200(1.05)^3 \frac{\left(1-\left(\frac{1}{1.05}\right)^3\right)}{12\left(1-\left(\frac{1}{1.05}\right)^{1/12}\right)} = 3884.69$$

   Hence the accumulation is $£3,884.69$

3. John Brown is repaying a loan of $£10,000$ by annual payments in arrears over a 10 year period. The interest rate is variable and initially the APR charged is 8%. After 3 years the APR is increased to 10% so that his annual payment has to be increased in line with the new interest rate. The term of the loan remains unaltered.

   (i) Find his initial repayment level. This is just $£P$ where $10000 = Pa_{\overline{10}|}$ and $i = 0.08$. Hence

$$P = \frac{10000\left(1-\frac{1}{1.08}\right)}{\frac{1}{1.08}\left(1-\left(\frac{1}{1.08}\right)^{10}\right)} = 1490.29$$

   Hence his initial repayment level is $£1,490.29$

   (ii) Find the amount of loan outstanding at the time of the interest rate change, i.e. when his third payment has just been made.

This MUST be calculated with the ORIGINAL interest rate of 8%. Either calculate the value at time 3 of the loan minus the repayments made (i.e. $\left(10000 - Pa_{\overline{3}|}\right)(1+i)^3$) or the value at time 3 of the remaining payments (i.e. $Pa_{\overline{7}|}$). I will do the latter calculation; both give approximately the same answer. The amount outstanding in pounds is

$$1490.29 \times \frac{\frac{1}{1.08}\left(1 - \left(\frac{1}{1.08}\right)^7\right)}{\left(1 - \frac{1}{1.08}\right)} = 7759.00$$

(iii) Find the new repayment level. The interest rate is now 10% for the remaining 7 years so you use $i = 0.1$. The new repayment is £P^* where the amount outstanding from (ii) is equal to $P^* a_{\overline{7}|}$. Hence

$$P^* = 7759 \times \frac{\left(1 - \frac{1}{1.1}\right)}{\frac{1}{1.1}\left(1 - \left(\frac{1}{1.1}\right)^7\right)} = 1593.75$$

4. The lifetime X has density function $f_X(x) = \frac{1}{5}$ for $0 < x < 5$.

(i) Find the survival function $S(x)$ for $0 \leq x \leq 5$.

$$S(x) = P(X > x) = \int_x^\infty f_X(x)dx = \int_x^5 \frac{1}{5}dx = 1 - \frac{x}{5}$$

(ii) If $K(x)$ is the curtate further lifetime of a life aged x, write $P(K(x) = k)$ in terms of the survival function and hence calculate $P(K(x) = k)$ and specify the range.

$P(K(x) = k) = \frac{S(x+k) - S(x+k+1)}{S(x)}$ is defined for all $x \geq 0$ for which $S(x) > 0$, i.e. for all $0 \leq x < 5$. Since no-one survives to 5, if you have lived x years already then $k < 5 - x$. Hence k takes integer values with $0 \leq k < (5 - x)$,

$$P(K(x) = k) = \frac{\left(1 - \frac{x+k}{5}\right) - \left(1 - \frac{x+k+1}{5}\right)}{\left(1 - \frac{x}{5}\right)} = \frac{1}{(5-x)}$$

Calculate $\mathsf{e}_x = E[K(x)]$ for $x = 0, 1, 2, 3, 4$.

Either do directly from the distribution of $K(x)$ so that

$$\mathsf{e}_x = E[K(x)] = \sum_{k=0}^{4-x} k \frac{1}{(5-x)} = \frac{1}{(5-x)} \frac{(4-x)(5-x)}{2} = \frac{4-x}{2}$$

Or use the result that $\mathsf{e}_x = \frac{1}{S(x)} \sum_{r=1}^{\infty} S(x+r)$. Note that here $S(x+r) = 0$ if $x + r \geq 5$ so that the upper end point of the range is effectively $4 - x$. Hence $\mathsf{e}_x = \frac{1}{(5-x)} \sum_{r=1}^{(4-x)} (5 - x - r)$. Put $k = 5 - x - r$ in the summation and you get the same sum as in the previous method. You can, of course also calculate either sum for each value of x.

Then $\mathsf{e}_0 = 2$, $\mathsf{e}_1 = 3/2$, $\mathsf{e}_2 = 1$, $\mathsf{e}_3 = 1/2$ and $\mathsf{e}_4 = 0$.

5 Use the life table ELT12 to obtain the following results:

(a) Calculate the probability that a life aged 20 survives to age 65.

$${}_{45}p_{20} = \frac{l_{65}}{l_{20}} = \frac{68490}{96293} = 0.7113$$

Hence the probability of survival is 0.7113

(b) Calculate the expected number of deaths by age 40 out of 1000 30-year olds.

Either: Scale ${}_{10}d_{30}$ by $\frac{1000}{l_{30}}$ to give

$$\frac{1000(l_{30}-l_{40})}{l_{30}} = \frac{1000(95265-93790)}{95265} = 15.4831$$

Or: Use the result that the number of deaths Y has binomial distribution with parameters 1000 and ${}_{10}q_{30}$ so that the expected number of deaths is $1000{}_{10}q_{30} = \frac{1000(l_{30}-l_{40})}{l_{30}}$ (the same as by first method).

Hence the expected number of deaths is 15.4831

(c) Show that ${}_{n|t}q_x = {}_np_x \times {}_tq_{x+n}$.

Either:

$${}_{n|t}q_x = \frac{l_{x+n}-l_{x+n+t}}{l_x} = \frac{l_{x+n}}{l_x}\frac{(l_{x+n}-l_{x+n+t})}{l_{x+n}} = {}_np_x \times {}_tq_{x+n}$$

Or: (harder)

$$\begin{aligned}
{}_{n|t}q_x &= P(n < T(x) \le n+t) \\
&= P(x+n < X \le x+n+t|X > x) \\
&= P(X \le x+n+t|X > x \text{ and } X > x+n)P(X > x+n|X > x) \\
&= P(X \le x+n+t|X > x+n)P(X > x+n|X > x) \\
&= P(T(x+n) \le t)P(T(x) > n) \\
&= {}_tq_{x+n} \times {}_np_x
\end{aligned}$$

(d) Let x be an integer and let $0 < t < 1$. Use linear interpolation of the survival function between integer ages to show that ${}_tq_x = t \times q_x$.

$${}_tq_x = \frac{l_x - l(x+t)}{l_x} \simeq \frac{l_x-((1-t)l_x+tl_{x+1})}{l_x} = \frac{t(l_x-l_{x+1})}{l_x} = t \times q_x$$

(e) Find the probability that a life who has just reached his 65^{th} birthday survives to 70 but dies within the next month.

We can use parts (c) and (d) to give

$${}_{5|(1/12)}q_{65} = {}_5p_{65} \times {}_{1/12}q_{70} \simeq \frac{l_{70}}{l_{65}} \times \frac{1}{12} \times q_{70} = \frac{54806}{68490} \times \frac{0.05566}{12} = 0.003712$$