Lecture's ~~§~~7 Handout (21/1/08)

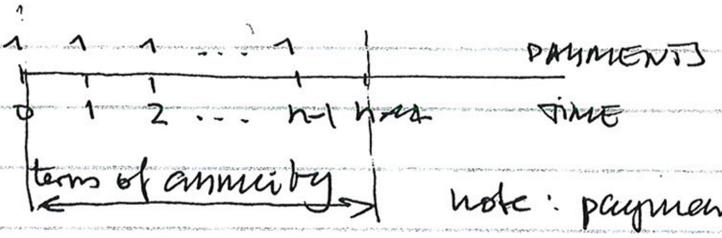
To recap from last ~~year~~ week

Discounting: the present value (P.V.) of one unit of money due in time t is v^t, where $v = 1/(1+i)$.
Indeed, v^t units of money invested at time $t=0$ will have the accumulated value of $v^t(1+i)^t = 1$ at time t

The present value of a discrete cash flow is the sum of the P.V.s of payments.

Examples of discrete cash flows — annuities (certain)
Annuities-due — payments made in advance
Immediate annuities — payments made in arrears

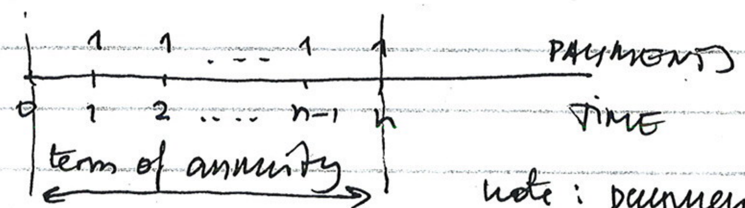
P.V. of annuity-due payable yearly at rate 1 units of money per annum for n years



$$\boxed{\ddot{a}_{\overline{n|}} = \frac{1-v^n}{1-v}}$$

note: payments in advance

P.V. of immediate annuity payable yearly at rate 1 u.m. per annum for n years



$$\boxed{a_{\overline{n|}} = v\,\frac{1-v^n}{1-v}}$$

note: payments in arrears

P.V. of annuity-due payable p-thly at rate 1 u.m. per annum for n years

$$\begin{array}{llllllll} 1/p & 1/p & 1/p \ldots & 1/p & \ldots & 1/p & & \text{PAYMENTS} \\ 0 & 1/p & 2/p \ldots & 1 & \ldots & n-1/p & n & \text{TIME} \end{array}$$

term of annuity

$$\boxed{\ddot{a}^{(p)}_{\overline{n|}} = \frac{1}{p}\,\frac{1-v^n}{1-v^{1/p}}}$$

Note: payments in advance, rate of pay 1 per year, hence each payment is $1/p$ units of money.

P.V. of immediate annuity payable p-thly at rate 1 u.m. per annum for n years

$$\begin{array}{llllllll} & 1/p & 1/p \ldots & 1/p & \ldots & 1/p & 1/p & \text{PAYMENT} \\ 0 & 1/p & 2/p \ldots & 1 & \ldots & n-1/p & n & \text{TIME} \end{array}$$

term of annuity

$$\boxed{a^{(p)}_{\overline{n|}} = \frac{1}{p}\,\frac{v^{1/p}(1-v^n)}{1-v^{1/p}}}$$

Note: payments in arrears, each payment is $1/p$ so that the total paid in one year is 1.

<u>Annuities paid in perpertuity</u> ($n=\infty$, payments go forever)

P.V. can be obtained from those for n-term annuities by letting $n\to\infty$. For example, the present value of an annuity-due payable monthly at rate 1 u.m. per year in perpertuity is

$$\ddot{a}^{(12)}_{\overline{\infty|}} = \lim_{n\to\infty} \ddot{a}^{(12)}_{\overline{n|}} = \lim_{n\to\infty} \frac{1}{12}\,\frac{1-v^n}{1-v^{1/12}} = \frac{1}{12}\,\frac{1}{1-v^{1/12}}$$

Note that $v^n \to 0$ as $n\to\infty$ because $v=\frac{1}{1+i}<1$.

P.V. of <u>continuous cash flows</u> can be obtained from those of p-thly annuities by letting $p\to\infty$. For example, Note that in the continuous limit ($p\to\infty$) there is no difference between paying in advance or in arrears, hence $\lim_{p\to\infty} \ddot{a}^{(p)}_{\overline{n|}} = \lim_{p\to\infty} a^{(p)}_{\overline{n|}}$

the present value of a continuous cash flow at rate $\rho(t)=1$ over n years is

$$\bar{a}_{\overline{n|}} = \lim_{p\to\infty} \ddot{a}^{(p)}_{\overline{n|}} = \lim_{p\to\infty} \frac{1}{p}\,\frac{1-v^n}{1-v^{1/p}}$$

$$= (1-v^n)\lim_{x\to 0} \frac{x}{1-v^{x}} \qquad [\text{by setting } x = 1/p]$$

$$= \frac{1-v^n}{-\ln v} \qquad [\text{by L'Hopital rule}]$$

For example, if $n=1$ then

$$\bar{a}_{\overline{1|}} = \frac{1-v}{-\ln v} = \frac{iv}{\delta} \qquad \left[\text{as } v=\frac{1}{1+i} \text{ and } (1+i)=e^{\delta}\right]$$

<u>Important</u>: ~~If payments~~ To find the present value of cash flows with yearly payment C, multiply $\ddot{a}_{\overline{n|}}$, $a_{\overline{n|}}$, $\ddot{a}^{(p)}_{\overline{n|}}$, $a^{(p)}_{\overline{n|}}$, $\bar{a}_{\overline{n|}}$ by C.

We say that two payments are *equivalent* if their present values coincide.

Five equivalent ways of paying interest on a loan of 1 unit of money over $[0,1]$:

	$t=0$	$t=\frac{1}{p}$	$t=\frac{2}{p}$	...	$t=\frac{p-1}{p}$	$t=1$	Time, t
(1)	d						
(2)	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	...	$\frac{d^{(p)}}{p}$		
(3)		$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	...	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	Payments
(4)						i	
(5)	$\longleftarrow$		Continuous payment at rate δ			$\longrightarrow$	

Notice that

- the present value of continuous payment at constant rate $\rho(t)=\delta$ over [0,1] (entry (5) in the table above) is $iv\frac{\rho}{\delta}\Big|_{\rho=\delta} = iv$,
- the present value of a single payment of i at time $t=1$ (entry (4) in the table) is iv,
- the present value of a single payment of d at time $t=0$ (entry (1) in the table) is d and $d=iv$.

Therefore (1) $\Leftrightarrow$(5)$\Leftrightarrow$(4).

Exercise: obtain (3)$\Leftrightarrow$(4)$\Leftrightarrow$(2).