Queen Mary, University of London

B.Sc. Examination by course units 2006/2007

MAS224 Actuarial Mathematics

Duration: 2 hours

Date and time: 9th May 2007, 2.30-4.30

Except for the award of a bare pass, only your best three questions will be counted. A sheet of formulae and certain life tables are provided for this examination.

Calculators ARE permitted in this examination, but no programming, graph plotting or algebraic facility may be used. The unauthorised use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the make and type of calculator used.

You should not start reading this paper until instructed to do so by the invigilator.

You must not remove the question paper from the examination room.

---

1. (a) State the relationship between the nominal rate of interest $i^{(p)}$ per annum when interest is compounded p-thly, the annual effective rates of interest i and the force of interest δ.

   (b) Find the annual effective rate of interest if the nominal rate of interest is 8% per annum when interest is compounded quarterly. How much interest should be paid in arrears for the use of £1000 for a period of (i) three months; (ii) one year; (iii) ten years?

   (c) Prove that $\lim_{p\to\infty} i^{(p)} = \delta$

   (d) Joe Bloggs took out a loan for £20,000 exactly five years ago and has just made a repayment. The loan is being repaid by monthly payments in arrears for a term of ten years. The APR charged on the loan is variable and was initially 10% and remained at that rate for five years. Find the initial level of monthly repayment. Find the amount currently outstanding on his loan.

   The APR has now changed to 12%.

   (i) Find the revised monthly repayment if the term of the loan remains unaltered.

   (ii) Find the revised monthly payment if the term remains unaltered but he is given a payment holiday of one year so that no payment is due during this period but interest will accrue. Once payments are resumed they are again paid monthly in arrears.

   (iii) Find the remaining term if the term for the loan is no longer fixed but the original repayment is retained. Obtain the value of the final payment made at the end of the new term.

2. Assume the mortality given in table ELT12.

   (a) Express the life table functions ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$, l_x and ${}_nd_x$ in terms of the survival function $S(t)$ and the radix l_0.

   (b) Calculate the probability that a life aged 18 survives to age 60.

   (c) Calculate the probability that a life aged 18 survives to age 60 but dies before age 65.

   (d) Calculate the expected number who die by age 60 out of 1000 individuals aged 18.

   (e) The complete (non-curtate) and curtate further lifetimes of a life aged x are denoted by $T(x)$ and $K(x)$ respectively. You may assume that $\mathring{e}_x = E[T(x)] = \frac{1}{S(x)}\int_0^\infty S(x+t)dt$ and $e_x = E[K(x)] = \frac{1}{l_x}\sum_{r=1}^\infty l_{x+r}$.

   Use linear interpolation on the survival function between integer ages to show that $\mathring{e}_x \simeq e_x + \frac{1}{2}$.

   (f) John Doe retires through ill health at age 63. With the exception of the first two years after retirement, his mortality is the mortality given in table ELT12. The chance he survives the first year after retirement is $\frac{1}{2}$ and the chance that he will then survive the following year is $\frac{3}{4}$.

   Derive John's complete non-curtate further expectation of life (i) at the time of his retirement; (ii) one year after retirement.

   Evaluate the probability that he survives to the normal retirement age of 65 but dies within the following year.

3. Assume a 4% annual interest rate and the mortality given by table A1967-70 select values.

   (a) Find the cost of a 20-year endowment assurance policy, with equal death and endowment benefits of £100,000, taken out by a life aged 30. The death benefit is payable at the end of the year of death.

   (b) Calculate the annual premium required to purchase the policy in part (a) if the premium is to be paid annually in advance during the term of the policy and contingent on survival.

   (c) Find the surrender value of the policy if the life assured surrenders the policy at age 40, just before the next payment is due.

   (d) Let Y be the present value of an n-year endowment assurance policy for a life aged x with death and endowment benefits of one unit, with the death benefit payable at the end of the year of death.

   Write Y in terms of $K(x)$, the number of complete further years lived by a life aged x.

   Show that $P(K(x) = k) = \frac{l_{x+k} - l_{x+k+1}}{l_x}$ for $k = 0, 1, 2, \ldots$

   Hence show that $E[Y] = 1 - d\frac{N_x - N_{x+n}}{D_x}$.

   Derive an expression for $Var(Y)$.

4. (a) Let X be the age-at-death of an individual and $S(x) = P(X > x)$ be the survival function. Show that the density function of $T(x)$, the complete further lifetime of a life aged x, is given by $f_{T(x)}(t) = \frac{-S'(x+t)}{S(x)}$ for $t > 0$.

   (b) Consider a whole life assurance policy for a life aged x with one unit of death benefit payable at the instant of death. Express the present value of the policy, Y, in terms of $T(x)$ and prove that $E[Y] = 1 - \frac{\delta}{D_x}\int_0^\infty D(x+t)dt$.

   (c) If $S(x) = e^{-\theta x}$ for $x \geq 0$, obtain the value of $E[Y]$ from part (b).

   (d) Now assume the mortality of table A1967-70 select values and an interest rate of 4%.

   John Smith purchased a whole life assurance from a Life Assurance Company on his 25th birthday, which he paid for by annual premiums in advance of £300 per year. The death benefit was payable at the instant of death. How much death benefit did this purchase?

   John Smith actually died on his 30th birthday, just before paying the next premium. Find the actual loss made by the Life Assurance Company on his policy at the time the death benefit was paid.

5. Consider females in a population in which there is no immigration or emigration. Let $n_x(t)$ be the expected number of females aged x in the population at time t. Females are assumed to have independent lifetimes and to reproduce independently. The probability that a female aged x at time t will survive to time $t+1$ is p_x and the expected number of female offspring she will produce in time t to $t+1$ is b_x.

   You may assume that, for $t = 0, 1, 2, \ldots$,

$$\begin{aligned} n_0(t+1) &= \sum_{x=0}^{\infty} b_x n_x(t) \\ n_{x+1}(t+1) &= p_x n_x(t) \text{ for } x = 0, 1, 2, \ldots \end{aligned}$$

(a) Suppose that this system of equations has a solution $n_x(t) \equiv n_x$. Show that $n_x = n_0 S(x)$ for $x = 0, 1, \ldots$ and $\sum_{x=0}^{\infty} b_x S(x) = 1$, where $S(x)$ is the survival function. Interpret this latter relation.

An isolated population has equal birth and death rates with the expected numbers in each age group constant over time. The life experience is that of table ELT12, so that $n_x = Al_x$ for a suitable scaling factor A. If $N(x)$ is the number aged x or more in the population, then you may assume that $N(x) \doteqdot Al_x\left(\frac{1}{2} + \mathring{e}_x\right)$. If the expected number of pensioners aged 60 or over in the population is 1,000, find the expected number of births each year and the expected total population size.

(b) Consider a population for which $p_x = p$ and $b_x = b$ for all x. Initially there are $N(0)$ individuals in the population. Obtain and solve the difference equation for $N(t)$, the expected population size at time t, where t is a non-negative integer. Determine the condition under which the expected population size is growing.

Let $r_x(t) = \frac{n_x(t)}{N(t)}$. Obtain equations for $r_x(t)$ and find $\lim_{t \to \infty} r_x(t)$.

(c) Animals in a population die before they reach age 3. Females produce offspring at ages 1 and 2; the expected number of offspring at these ages is 1 and 2 respectively. The probability that a female survives her first year of life is $\frac{2}{3}$ and the probability that a one-year old female survives to age 2 is $\frac{1}{2}$.

Write down the Leslie matrix. If initially there are 100 individuals in each age group x, for $x = 0, 1, 2$, find $n_x(t)$ for $x = 0, 1, 2$ and $t = 1, 2$. Will the mean numbers in the age groups stabilise over time? Justify your answer.

A list of selected formulae are given below.

Present values of annuities certain:

$$\ddot{a}_{\overline{n}|} = \frac{1-V^n}{1-V} \qquad a_{\overline{n}|} = V\,\ddot{a}_{\overline{n}|}$$

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{1}{p}\left[\frac{1-V^n}{1-V^{\frac{1}{p}}}\right] \qquad a^{(p)}_{\overline{n}|} = V^{\frac{1}{p}}\,\ddot{a}^{(p)}_{\overline{n}|}.$$

Expected present values of life annuities:

$$\ddot{a}_x = \frac{N_x}{D_x} \qquad \ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x} \qquad a_x = \ddot{a}_x - 1$$

$$\ddot{a}^{(p)}_x \simeq \ddot{a}_x - \frac{p-1}{2p} \qquad a^{(p)}_x \simeq a_x + \frac{p-1}{2p} \qquad a^{(p)}_x = \ddot{a}^{(p)}_x - \frac{1}{p}$$

Conversion relationships:

$$\bar{A}_x = 1 - \delta\bar{a}_x$$

$$A_x = 1 - d\,\ddot{a}_x$$

$$A^{(p)}_x = 1 - d^{(p)}\,\ddot{a}^{(p)}_x$$

$$A^{(p)}_{x:\overline{n}|} = 1 - d^{(p)}\,\ddot{a}^{(p)}_{x:\overline{n}|}$$

End of examination. Three pages of tables follow.