Spectral determinants of complex random matrices

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J. Phys. A: Math. Theor. **40** (2007) 669-699

and math-ph/0602032

Spectral Determinants

Motivation: - complex eigenvalues

$\langle |\det(zI - W)|^2 \rangle_W$ related to eigv. distr. of W (sometimes explicitly)

Example 1: Gaussian ensembles $P(W_N, W_N^\dagger) \propto e^{-\operatorname{tr} W_N W_N^\dagger}$

Mean density $\rho_N(x, y)$ of complex egenvalues $z = x + iy$:

$$\rho_N(x, y) \propto e^{-|z|^2} \left\langle |\det(zI_{N-1} - W_{N-1})|^2 \right\rangle_{W_{N-1}} \qquad \text{(complex } W\text{)}$$

$$\rho_N(x, y) \propto y e^{-(x^2 - y^2)} \operatorname{erfc}(y) \left\langle |\det(zI_{N-2} - W_{N-2})|^2 \right\rangle_{W_{N-2}} \qquad \text{(real } W\text{)}$$

but mean density of real eigvs of real matrices is prop. to mean absolute value of spectral determinant,

$$\rho_N(x) \propto e^{-x^2} \langle |\det(xI_{N-1} - W_{N-1})| \rangle_{W_{N-1}}$$

Eigenvalue corr. fncs are expressed in terms of higher moments of spectral dets.

[Ginibre ’64, Lehmann & Sommers ’91, Edelman ’93, Edelman, Kostlan & Shub ’94]

Spectral Determinants

Example 2: Finite rank deviations from Hermiticity or unitarity. E.g.,

If $W_N(\gamma) = R_N U_N$ where U_N is CUE and $R_N = \mathrm{diag}(\sqrt{1-\gamma}, 1, \ldots, 1)$ (rank-one deviations from CUE) then

$$\rho_N(x,y) = \frac{N-1}{\pi\gamma|z|^2}\left(\frac{\tilde{\gamma}}{\gamma}\right)^{N-2} \langle|\det(zI_{N-1} - W_{N-1}(\tilde{\gamma}))|^2\rangle_{U_{N-1}},$$

where $\tilde{\gamma} = \frac{|z|^2+\gamma-1}{|z|^2}$. Note that $\gamma = 1$ corresponds to subunitary matrices (delete 1st row & column). In this case $\tilde{\gamma} = \gamma = 1$.

If $W_N(\gamma) = H_N + i\Gamma_N$, where H_N is GUE and $\Gamma_N = \mathrm{diag}(\gamma, 0, \ldots, 0)$ (rank-one deviations from GUE), then

$$\rho_N(x,y) = r_{N,\gamma}(x,y)\langle|\det(zI_{N-1} - W_{N-1}(\tilde{\gamma}))|^2\rangle_{H_{N-1}}$$

where $\tilde{\gamma} = \gamma - y$.

[Fyodorov & K ’99, Życzkowski K & Sommers ’02, Fyodorov & Sommers ’03]

Spectral Determinants

Previous examples are special. In general, one would recover the mean density of eigenvalues from the mean fractional (absolute) moments of the spectral dets

$$G(x,y) = \langle |\det(zI - W)|^{2s} \rangle_W$$

Because of the singularities, some sort of regularization might be desirable, e.g.,

$$\langle |\det(zI - W)(zI - W)^\dagger + \varepsilon I|^{2s} \rangle_W$$

Unfortunately, we can handle integer moments only, $G(x,y)$ for integer s.

Note that if the distribution of W is invariant then

$$\langle |\det(zI - W)|^{2s} \rangle_W = \left\langle \int_{U(N)} |\det(zI - WU)|^{2s} dU \right\rangle_W$$

In view of this, we consider the class of matrices $W = AU$ where A is fixed and U is chosen at random from the unitary group $U(N)$. We shall see that the integration over U (the 'angular' part of W) reduces non-Hermitian problem (moments of the spectral determinants) to a Hermitian one.

Angular integrals

For any two $N \times N$ matrices A and B

$$\int_{U(N)} \det(I - AU)^m \det(I - U^\dagger B^\dagger)^n dU \propto$$

$$\int_{\mathbb{C}^{n\times m}} \frac{\det(I + Q^\dagger Q \otimes B^\dagger A)}{\det(I + Q^\dagger Q)^{N+n+m}} \, dQ, \quad m, n \geq 1.$$

The integration on the RHS is over rectangular matrices Q of size $n \times m$.

If $A^\dagger A < I_N$ and $B^\dagger B < I_N$ then

$$\int_{U(N)} \frac{dU}{\det(I - AU)^m \det(I - U^\dagger B^\dagger)^n} = \int_{Q^\dagger Q \leq I} \frac{d\rho_{N,n\times m}(Q)}{\det(I - Q^\dagger Q \otimes B^\dagger A)},$$

$$1 \leq m, n \leq N,$$

where $d\rho_{N,n\times m}$ is the push-forward of the Haar measure under the truncation $U \mapsto Q$

Note that if $N \geq n + m$ then $d\rho_{N,n\times m}(Q) \propto \det(I - Q^\dagger Q)^{N-m-n} dQ$ [Friedman & Mello '85; also Neretin '02, Fyodorov & Sommers '03, Forrester '06]

Schur function expansions and CFT

The above integration formulas can be proved by making use of either the Schur function expansions or Zirnbauer's Colour-Flavour Transformation (Zirnbauer '96), and in fact these two approaches are equivalent. The equivalence comes in the form of another pair of integral identities where s_λ are Schur functions:

- Fermionic case. For integer $N \geq 0$

$$\int_{\mathbb{C}^{n\times m}} \frac{s_\lambda(Q^\dagger Q)}{\det(I + Q^\dagger Q)^{N+m+n}}\, dQ = const. \frac{s_\lambda(I_m) s_\lambda(I_n)}{s_{\lambda'}(I_N)} \qquad \text{(UIF)}$$

- Bosonic case. For integer $N \geq n, m$

$$\int s_\lambda(Q^\dagger Q) d\rho_{N,n\times m}(Q) = \frac{s_\lambda(I_m) s_\lambda(I_n)}{s_\lambda(I_N)} \qquad \text{(UIB)}$$

(UIB) is a corollary of the invariance of $d\rho_{N,n\times m}$, (UIF) seems to be new.

(UIB) implies the bCFT via Schur functions expansions, and vice versa. We can only show that (UIF) is equiv. to the fCFT in a particular case (corresponding to $\langle |\det(I + AU)|^2 \rangle_U$.

Selberg-type integrals

(UIB) can also be obtained from the Selberg type integral ($J_\lambda^{1/\gamma}$ are Jack polynomials)

$$\int_0^1 \cdots \int_0^1 J_\lambda^{\frac{1}{\gamma}}(x_1, \ldots, x_m) \prod_{j=1}^{m} x_j^{p-1}(1-x_j)^{q-1} \prod_{1\le i<j\le m} |x_i - x_j|^{2\gamma} \prod_{j=1}^{m} dx_j$$

$$= \quad J_\lambda^{\frac{1}{\gamma}}(1_m) \prod_{i=1}^{m} \frac{\Gamma(i\gamma+1)\Gamma(\lambda_i+p+\gamma(m-i))\Gamma(q+\gamma(m-i))}{\Gamma(1+\gamma)\Gamma(\lambda_i+p+q+\gamma(2m-i-1))}.$$

evaluated by Kadell '88 (for $J_\lambda^1 = s_\lambda$), Kadell '97 (general), Yan '92, Kaneko '93.

$< J_\lambda^{\frac{1}{\gamma}} >$ in the fermionic case yet to be evaluated for arbitrary γ which is an interesting open problem. Known cases $\gamma = 1, 2$.

$\gamma = 1$: (Schur functions) the integral in both cases, can be evaluated by reducing it to binomial determinants (Fyodorov & K '06).

$\gamma = 1/2$ (zonal polynomials): the integral was evaluated by Constantine '63 in the bosonic case and his calculation can be extended to the fermionic case.

Applications: Feinberg-Zee single ring theorem

Consider random matrices $W \in \mathbb{C}^{N\times N}$ with inv. matrix distr. $e^{-N\,\mathrm{Tr}\,V(W^*W)}dW$. Note that joint pdf of eigenvalues is only known for the Ginibre ensemble ($V(t) = t$).

In view of unitary invariance,

$$\langle|\det(zI - W)|^{2m}\rangle_W \propto \int_{\mathbb{C}^{m\times m}} \frac{\langle\det(|z|^2 I + Q^\dagger Q \otimes W^\dagger W)\rangle_W}{\det(I + Q^\dagger Q)^{N+2m}}\,(dQ)$$

Thus, integration over the angular part of W can be traded for an average over $m \times m$ matrices Q - Jacobi ensemble. Advantage - now have Hermitian matrices $W^\dagger W$, can apply orthogonal polynomial technique, etc. Structure - Hankel determinants. Matrix elements are integrals involving orthogonal polynomials.

Also advantageous for small values of m. E.g., $m = 1$

$$\left\langle|\det(zI - W)|^2\right\rangle_W = (N+1)\int_0^{+\infty} \frac{\left\langle\det(I|z|^2 + tW^\dagger W)\right\rangle_W}{(1+t)^{n+2}}\,dt$$

Applications: Feinberg-Zee single ring theorem

$\left\langle \det(I|z|^2 + tW^\dagger W)\right\rangle_W$ can be evaluated asymptotically for large N in terms of the eigv. distribution $d\sigma(\lambda)$ of the Hermitian matrices $W^\dagger W$, yielding

$$\left\langle |\det(zI - W)|^2 \right\rangle_W = \exp[N\Phi(x,y) + o(N)]$$

$$\Phi(x,y) = \begin{cases} \log|z|^2 & \text{if } |z| > m_1 = \int \lambda d\sigma(\lambda), \\ \int_0^\infty \log\lambda d\sigma(\lambda) & \text{if } 1/|z| > m_{-1} = \int \frac{d\sigma(\lambda)}{\lambda}, \\ |z|^2 + \int_0^\infty \log\frac{\lambda+t_0}{|z|^2+t_0} d\sigma(\lambda) & \text{if } 1/m_{-1} < |z| < m_1 \end{cases}$$

where t_0 is the (unique) solution of $\int_0^\infty \frac{d\sigma(\lambda)}{\lambda+t} = \frac{1}{|z|^2+t}$.

Strong self-averaging (Berezin '73)

$$\lim_{N\to\infty}\left\langle \frac{1}{N}\log|\det(zI - W)|^2 \right\rangle_W \stackrel{(?)}{=} \lim_{N\to\infty}\frac{1}{N}\log\left\langle |\det(zI - W)|^2\right\rangle_W$$

Yes for GUE (Berezin '73). Yes for Ginibre (by direct computation $\Phi(x,y) = |z|^2 - 1$). Yes beyond Ginibre as $\Delta\Phi$ agrees with mean eigv density of W found by Feinberg & Zee '97.

Regularised inverse determinants

$$R_\varepsilon(A^*A) = \int_{U(N)} \frac{dU}{\det[(I+AU)(I+AU)^* + \varepsilon^2 I]^m}$$

Non-trivial even for $m = 1$. Direct application of bCFT runs into a problem (diverging integrals). Schur functions do not help. A deformed version of CFT helps. Expression for $R_\varepsilon(A^*A)$ in the simplest case $m = 1$:

$$\frac{N-1}{2\pi i}\int_0^1 (1-t)^{N-2} dt \int_{-\infty}^{+\infty} \frac{ds}{s} \frac{1}{\det\left[A^\dagger A + (\varepsilon^2 - t)\, I - i\varepsilon\sqrt{t}\left(s+\frac{1}{s}\right) I\right]}.$$

If the eigenvalues a_j^2 of $A^\dagger A$ are distinct then, in the limit $\varepsilon \to 0$, this integral can be evaluated: $-c_N(z)\log\varepsilon^2 + d_N(z) + O(\varepsilon)$,

$$c_N(z) = (N-1)\sum_{j=1}^{N} (1-|z|^2 a_j^2)^{N-2}\,\theta(1-|z|^2 a_j^2) \prod_{k\neq j} \frac{1}{|z|^2(a_k^2 - a_j^2)}$$

where θ is Heaviside's step fnc. For $\lambda_{min}(A^\dagger A) \leq \frac{1}{|z|^2} \leq \lambda_{max}(A^\dagger A)$ have log-singularity ($c_N(z) \neq 0$).

Conclusions

- moments of spectral determinants is an interesting object, various links to truncations of random unitary matrices, CFT, Selber-type integrals, Berezin reproducing kernels (Berezin '75)
- stochastic Horn problem (singular values $\rightsquigarrow$ eigenvalues) for spectral determinants can be solved by two equivalent methods: Schur function expansions or CFT.
- Feinberg-Zee's ring density reproduced (but not proved); have conjecture:

$$\frac{1}{N} < \log\det >= \frac{1}{N} \log < \det > \text{ (strong non-Hermiticity)}$$

- fractional moments or averages of ratios of spectral dets wanted

$$\text{mean eigv density.} = \lim_{\varepsilon\to 0} \frac{\partial}{\partial \overline{z}} \lim_{z\to\zeta} \frac{\partial}{\partial \overline{\zeta}} \left\langle \frac{\det[\varepsilon^2 I + (zI - W)(zI - W)^\dagger)]}{\det[\varepsilon^2 I + (\zeta I - W)(\zeta I - W)^\dagger)]} \right\rangle$$

- other classical groups?

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