SCHOOL OF MATHEMATICAL SCIENCES

PROPOSAL FOR CHANGES IN THE LINEAR ALGEBRA STREAM
(BK, 3 April 2007)

Log of changes to the previously circulated (by e-mail) draft:

BJ's suggestion to make diagonalisation a separate topic is added – 26/03/07
Linear equations content of Geometry I added (thanks to LHS) – 3/4/07
Fundamental Theorem of Linear Algebra corrected (thanks to FJW) – 3/4/07

I would like to put forward a proposal for changes in teaching the Linear Algebra stream. Currently it consists of three modules MAS114 Geometry I, MAS212 Linear Algebra I and MAS317 Linear Algebra II offered in semesters 1, 3 and 5 respectively.

Part I
Rational for change: There is an opinion that Linear Algebra I – the key module in the stream - is too abstract to serve the needs of the statistics and applied streams in the UG maths programme and is in need of adjustment. Some of the topics covered in Linear Algebra II (e.g. Gram-Schmidt orthogonalisation procedure) come in too late and could be moved to Linear Algebra I.

In 2006-7 Geometry I was changed to place more emphasis on vectors and matrices (geometric approach). The matrices content of Geometry I is summarised below in Part II. Apart from matrix addition and multiplication the concepts are developed in the context of ***R***^*2* and ***R***^*3* only, and from the geometric viewpoint. It is expected that on leaving Geometry I students are familiar with the concept of vector and matrix addition and multiplication. They are also expected to be familiar with determinants and their properties and with the concept of linear transformations and relation to matrices. Some familiarity with eigenvalues and eigenvectors is also expected.

What are the proposed changes to Linear Algebra I: remove definition of a field and examples to Linear Algebra II; make Gaussian elimination the starting point; spend more time on matrix algebra; develop the concept of vector space for ***R***^n and follow it with real and complex inner product spaces; spend more time on orthogonality, orthogonal bases and orthogonal projections culminating in the geometric interpretation of the solution to the least squares problem and Gram-Schmidt orthogonalisation. Spend time on motivating examples (Markov chains, regression, adjacency matrix (graphs), etc) and on various examples of vector and inner product spaces and subspaces.

The starred and crossed items in the list of learning outcomes, appended at the end of Part III form the current LO for Linear Algebra I (references to fields have been omitted). I propose to retain the starred items and drop the crossed ones. There are also additional items to reflect the proposed changes.

It is proposed to retain the style of Linear Algebra I – a mixture of methods and proofs. As far as I can see the included proofs, with a few exceptions, are short and transparent provided one understands the basics. Given that the students were exposed to the concept of proof in Probability I and Geometry I the complexity of proofs is just about right. Students who took Introduction to Algebra are expected to cope better. The number of proofs is a concern. It is important to retain proofs to make the course intellectually stimulating for good students and to provide mid-ability students with a mean to test their understanding (wishful thinking, perhaps).

With one or two exceptions, the proposed Linear Algebra core itemised below in Part III contains the essential material, which in my view should be taught to all students irrespectively of their specialisation. My estimate is that it would take about 28 lectures to cover it. Planning for spare lectures, it leaves about 3 lectures for extras and extensions. There are several views on what could be taught in this space. Those that I know are summarised in items 9-12 in Part III. Given that the core is already overloaded with concepts, my recommendation is to go for item 9 – enhance core material by examples. Bill's Jackson suggestion to make diagonalisation a separate topic, see item 13 is not incompatible with this. Other views are welcome.

I asked Francis Wright, the current LA I course organiser, for his views on the current Linear Algebra I. Here is what he has to say: " … Linear Algebra I is probably too abstract for most of our students. It might make sense to replace it with an "applied" module on matrices in Semester 3 followed by a "pure" module on vector spaces in Semester 4. The former could be recipe-orientated with few proofs, whereas the latter could be more rigorous and contain a fair number of theorems and proofs. The module on matrices could also include some numerical computation. Then Introduction to Numerical Computing could build on it, which might make that module more intelligible and attractive to our students."

Though it is possible to follow Francis' suggestion, it will take some manoeuvring, especially in semester 4, and may be disadvantageous for the pure maths stream in the UG programme and may lead to a split of LA I into two separate courses, one for joint degrees and one for single honours. This is undesirable (think about maths and stats, is it a joint degree or not). Again, views are welcome.

Implications of the proposed changes for follow up courses:

Linear Algebra II: it is desirable to retain LA I as the sole prerequisite, minor changes will be required:

- Include fields, arithmetic of finite fields, examples.
- Spend more time in "Revision of vector spaces" to introduce vector spaces over general fields.
- Remove "Orthogonality, the Gram-Schmidt orthogonalisation process, orthogonal projections".

Depending on whether there was time to cover eigenvalues of symmetric matrices in LA I, one can think of including item 11 (part or in full).

Coding Theory: change the prerequisites from "Linear Algebra I" to "Linear Algebra I and one of Linear Algebra II or Introduction to Algebra"

Cryptography: same change as for Coding Theory

Part II
Matrices content of MAS114 Geometry I, 2006-7 version

1) Rectangular matrices: laws of addition, scalar multiplication and multiplication.
2) Invertible matrices (*AB=BA=I*), uniqueness of matrix inverse
3) Determinants (*2x2* and *3x3* only)
   a) *2x2*: definition, verification of *det (AB)=det(A)det(B)*, explicit form of matrix inverse, geometric interpretation

b) *3x3*: introduced as the mixed product of columns (this yields immediately by the Laplace expansion and properties relative to elementary transformations on columns); *det(AB)=det (A) det (B)* stated but not verified; evaluation of dets with the help of elementary transformations; geometric meaning.

4) Vector space ***R****^n* (row- and column-vectors).
5) Linear equations in 2 and 3 variables: solution by reduction to echelon form
6) Linear transformations
   a) Definition (general), examples (reflection, rotation, stretches and shears – all *2x2*)
   b) Matrices as linear transformations
   c) Matrices of linear transformations, composition of linear transformations and matrix multiplication
7) Eigenvalues and eigenvectors:
   a) definition (general), characteristic polynomial (*2x2* and *3x3*), eigenvalues as roots of characteristic polynomial (stated for *2x2* and *3x3* but proved for *2x2* only)
   b) examples of finding eigenvalues and eigenvectors (*2x2* and triangular *3x3* matrices)
   c) eigenvalues and eigenvectors of reflections, rotations, stretches and shears in the plane, reflections in the 3d.

Part III
MAS212 Linear Algebra I – a possible version for 2007-8 – 33 lectures, 11 exercise classes.

Course text – Leon: Linear Algebra with Applications (7th edition), Pearson, £49.99
Other texts – Lipschutz & Lipson: Linear Algebra 3rd ed., Schaum Outlines series, £9.99

Core – 28 lectures + 2 spare (this leaves 3 lectures for extras)

1. Systems of linear equations - 3 lectures
   a) Consistent and inconsistent systems, equivalent systems, three types of elementary row operations.
   b) Gaussian elimination. Row echelon form, existence and non-uniqueness. Reduced row echelon form (uniqueness stated but not proved).
   c) Solution by reducing the augmented matrix to the (reduced) row echelon form. Solutions in vector form.
   d) Underdetermined systems (if consistent then has infinitely many solutions). Underdetermined homogeneous systems (always have infinitely many solutions). Geometric interpretation. Overdetermined systems (curve fitting, least-squares as examples).
   e) The special case n=m: a system is consistent and has a unique solution if and only if the corresponding homogeneous system has only zero solution.

      Drill: verify whether a given system is consistent or not, if consistent find all solutions, write solutions in vector form.

2. Matrix algebra - 3 lectures
   a) Revision of matrix addition/multiplication and matrix inverse (Geometry I). Matrix transpose, transpose of matrix product (via sigma-notation), transpose of matrix inverse.
   b) Special types of square matrices: diagonal (left and right multiplication by), lower/upper-triangular (addition and multiplication), unit lower/upper-triangular (form a group in the language of MAS117, inverse for *2x2* only), permutation (group), orthogonal (group).

c) Linear systems in matrix notation, including (1e) stated as "*A* is invertible if and only if $Ax=0$ has no non-zero solutions". Ax as a linear combination of the columns of A. Consistency Theorem for linear systems ($Ax=b$ is consistent if and only if b can be written as a linear combination of the columns of A). Elementary matrices and their inverses, row operations as multiplication to the left by elementary matrices.
d) Reduced row echelon form for square matrices: A is invertible if and only if its RREF is an identity matrix. Matrix A is invertible if and only if it is a product of elementary matrices. Finding matrix inverse by Gaussian elimination. (Showing that the inverse of a unit triangular matrix is unit triangular?)

> Drill: operation on matrices numerically and algebraically (addition, multiplication, transposition), calculating inverse matrices by Gaussian elimination.

3. Determinants - 3 lectures
   a) Inductive definition. Cofactors and row/column expansions, determinants of triangular matrices. Determinant of a matrix transpose.
   b) Determinants and elementary row/column transformations
   c) Determinant of matrix product (proof via elementary matrices)
   d) Equivalence of (i) det A=0, (ii) A is singular, (iii) Ax=0 for a non-zero x, and (iv) a non-trivial linear combination of columns/rows of A vanishes.
   e) Adj(A), matrix inverse in terms of adjoint. Cramer's rules for solution of Ax=b.

> Drill: evaluation of determinants with the help of row/column expansions and elementary operations on rows and columns.

4. Vector spaces (over ***R*** and ***C***) - 6 lectures
   a) Definition, examples: ***R***^*n*, ***R***^{*n*x*m*}, polynomials of degree <=*n*, the null space of a matrix, solutions of linear homogeneous differential equations (2^{nd} order will do), convergent series, functions with at least n derivatives.
   b) Subspaces: definition and examples to follow examples in (a).
   c) Linear combinations of vectors. Span of a (finite) set of vectors. Spanning sets for a vector space/subspace.
   d) Linear independence. Linear independence of columns for invertible matrices. Linear independence of functions, Wronskian.
   e) Basis and dimension of a vector space. Change of basis, transition matrix.
   f) Row and columns spaces of a matrix. Rank of a matrix. The null space and rank of matrix. The Rank-Nullity Theorem. The Kronecker-Capelli Theorem ($Ax=b$ is consistent if and only if A and $(A|b)$ have the same rank, i.e. b lies in the column space). Rank of matrix products.

> Drill: verifying axioms of vector space, verifying whether a given vector belongs to a subspace, verifying linear independence of vectors, verifying whether a given set of vectors forms a basis, find the coordinates of a vector with respect to a basis, calculating transition matrices, finding the rank of a matrix.

5. Linear Transformations – 3 lectures
   a) Definition and examples. The kernel and image of a linear transformation. Matrix representations of linear transformations. The Rank-Nullity Theorem in terms of linear transformations.

b) The law of change of matrix representation under the change of basis. Similarity.

Drill: verify whether a transformation is linear or not, calculate matrices of linear transformations with respect to a given basis.

================== MID TERM TEST ======================

6. Orthogonality in ***R***^*n* – 3 lectures
   a) Scalar product in ***R***^*n*: definition and properties. Euclidian distance between two vectors, length of a vector. The Cauchy-Schwarz inequality. Perpendicular vectors. Pythagoras Theorem. Orthogonal projection onto a vector.
   b) Orthogonal subspaces. Orthogonal complements. Orthogonal sums of subspaces. The Fundamental Theorem of Linear Algebra (the nullspace of A is orthogonal to the column space of the transpose to A and the dimensions add up to the number of columns of A) The Fredholm alternative for (finite) matrices.
   c) Orthogonal projection onto a subspace and Least Squares problems.

Drill: Calculate the inner product of two vectors and determine whether the vectors are (a) orthogonal, (b) orthonormal. Find vector projections onto (i) a vector, (ii) subspace. (Given a set *S* of vectors and a vector *f*, find the vector in the span of *S* closest to *f*).

7. Real inner product spaces – 3 lectures
   a) Definition of the inner product and examples of inner product spaces. The Cauchy-Schwarz inequality. Euclidian norm of a vector, distance.
   b) Orthonormal sets, bases. The Parseval identity. Orthogonal matrices as transition matrices form one orthonormal basis to another.
   c) The Gram-Schmidt orthogonalisation process and the *QR* factorisation. (Least squares again)

Drill: Same as above but for inner product spaces. Apply the Gram-Schmidt orthogonalisation process. Determine bases for the row and column spaces of a matrix and its transpose.

8. Eigenvalues and Eigenvectors – 4 lectures
   a) Definition, the characteristic polynomial, algebraic multiplicity, geometric multiplicity. Examples.
   b) Complex eigenvalues of real matrices. Complex inner product spaces. Fundamental theorem of algebra (without proof). Eigenvectors of real matrices.
   c) Eigenvalues and eigenvectors of special classes of matrices (real symmetric, real orthogonal, Hermitian, unitary). Orthogonal diagonalisation of a real symmetric matrix.
   d) Similarity: distinct eigenvalues and diagonalisation.

Drill: Calculate eigenvalues and eigenvectors of a square matrix. Reduce a matrix with distinct eigenvalues to a diagonal matrix by a similarity transformation. Calculate the eigenvalues and eigenvectors of a real symmetric matrix and transform it to diagonal matrix.

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Extras – 3 lectures

9. Enhancing core material by examples (Markov chains: n-step probabilities, eigenvectors and steady states, multivariate regression, adjacency matrix and counting the number of walks between two vertices of a graph, Leontieff model, Leslie population model (overlaps with Actuarial Maths), web searches and Google page rank algorithm, etc). Perron-Frobenius Theorem (without proof) if Markov chains and Leslie models are selected.

OR

10. Introduction to vector spaces over general fields. Finite field arithmetic, applications to coding and other examples. E.g. (thanks to **Robert Johnson**) applications to extremal set theory. Simplest example: How many subsets of $\{1,2,..,n\}$ can we have so that each of them has odd size and the intersection of any two has even size? One-line answer: At most n. Regard the sets as vectors in $\boldsymbol{Z}_2^n$ in the obvious way. The conditions meant they are orthogonal. So LI. So at most n. Obviously, this is more of an example sheet question that a serious theorem but there are extensions by Frankl and Wilson which give much more general results using nothing more than the dimension of various vector spaces of polynomials.

OR

11. The spectral theorem for real symmetric and Hermitian matrices. Schur decomposition for complex matrices (any complex matrix with distinct eigenvalues is unitary equivalent to a triangular). Singular value decomposition

OR

12. Introduction to the basic ideas of Linear Programming

    Plan (thanks to **Bill Jackson**)

    1. Solving LP problems graphically in 2-dim.: introduce the ideas of convex sets, extreme point, and the fact that the max occurs at an extreme point.
    2. Extend the ideas of convex sets and extreme points to $\boldsymbol{R}^n$. characterise extreme points of $\{x\in \boldsymbol{R}^n : Ax\leq b, x\geq 0\}$ as basic feasible solutions of $A'x'=0$ ie associated system of equalities we obtain by adding slack variables.

    3. Describe simplex algorithm for solving $\{x\in \boldsymbol{R}^n : Ax\leq b, x\geq 0\}$ in $\boldsymbol{R}^n$ and give an example in $\boldsymbol{R}^3$.

       cw
       q1 solve an lp problem graphically in $\boldsymbol{R}^2$
       q2 sove same problem using simplex algorithm
       q3 solve a problem in $\boldsymbol{R}^3$
       q4 something on formulating a `real life problem' as a LP problem.

13. More time spent on diagonalisation: Plan (thanks to **Bill Jackson**)

    i) Similarity and diagonalisation. Characterisation of diagonalisable matrices (sum of dimensions of eigenspaces is equal to order of matrix). Corollary for distinct eigenvalues. Example of 2x2 real matrix which is diagonalisable over C but not R.

Applications of diagonalisation: calculating powers of a matrix, markov chains, solving systems of differential equations and/or recurrence relations...

ii) Orthogonal diagonalisation of real symmetric matrices. Applications: quadratic forms, conics, etc.

Leaning Outcomes of Linear Algebra I:

Students passing this module should be able to do the following:

a) Solve a system of linear equations with the help of Gaussian elimination and write the solution in vector form.

b) * Multiply matrices, including rectangular, and calculate the transpose of a matrix where the entries are either scalar or algebraic expressions.

c) * Use algebraic equations *A(B + C) = AB + AC, (AB)^t = B^tA^t* etc., both with letters for matrices and with examples of matrices whose entries are either scalars or algebraic expressions.

d) * Determine whether a given matrix is invertible or not. Calculate the inverse of an invertible matrix.

e) ~~Given a square matrix *M*, use reduction to echelon form to find invertible matrices *P*, *Q* such that *PMQ* is diagonal;~~ -- KEEP THIS?

f) * Given a set *S* of row vectors and a row vector *v*, determine whether or not *v* is in the subspace spanned by *S*.

g) Determine whether given vectors (i) are linear independent (ii) form a basis for a vector space.

h) * Find the coordinates of an element in a vector space with respect to a given ordered basis.

i) Calculate the transition matrix corresponding to a change of basis in a vector space.

~~j) given *m* row vectors of length *n* over ***R***, where *m* < *n*, find a maximal linearly independent subset and expand it to a linearly independent set of cardinality *n*;~~

k) * Calculate the rank of a matrix.

l) Given a mapping from one vector space to another, verify whether it is linear or not.

m) * Given a linear mapping from one vector space to another, calculate the matrix of the mapping respect to given bases.

n) * Calculate the inner product of two vectors and determine whether the vectors are (a) orthogonal, (b) orthonormal.

o) Find vector projections onto (i) a vector, (ii) subspace. (Given a set *S* of vectors and a vector *f*, find the vector in the span of *S* closest to *f*).

p) Be able to apply the Gram-Schmidt orthogonalisation process.

q) Determine bases for the row and column spaces of a matrix and its transpose.

r) * Calculate eigenvalues and eigenvectors of a square matrix.

s) * Given a square matrix *M* over ***R*** with distinct eigenvalues, find an invertible matrix *P* such that *(P^{-1})MP* is diagonal.

t) * Given a symmetric matrix *M* over ***R***, find an orthogonal matrix *P* such that *(P^t)MP* is diagonal.