Schur function expansions and some matrix integrals

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We are interested $\int_{U(n)} |\det(zI - AU)|^{2m} d\mu_H(U)$.

Motivation - complex eigenvalues.

$\langle|\det(zI - W)|^2\rangle_W$ related to eigv. distr. of W (sometimes explicitly). E.g. Ginibre ensembles, finite rank deviations from Hermiticity or unitarity, in particular truncations of random unitary matrices.

The first moment, $\langle|\det(zI - AU)|^2\rangle_U$ can be handled with bare hands. Can work with diagonal A.

Write $|\det(zI - AU)|^2 = \det(zI - AU)\det(zI - AU)^*$ and expand each det in powers of z.

$$\begin{aligned}
\det(zI - AU) &= \sum_{r=0}^{n}(-1)^r z^{n-r} e_r(AU) \\
&= \sum_{r=0}^{n}(-1)^r z^{n-r} \sum_{1\le i_1<\ldots<i_r\le n} a_{i_1}\cdots a_{i_r} U(i_1,\ldots i_r),
\end{aligned}$$

Here the $e_r(AU)$ are elementary symmetric fncs of eigvs of AU and $U(i_1,\ldots i_r)$ are the principal minors of U.

By the invariance of the Haar meas, the cross product terms vanish and

$< \det(zI - AU)\det(zI - AU)^* >_U =$

$$\sum_{r=0}^n |z|^{2(n-r)} \sum_{1\le i_1<\ldots<i_r\le n} a_{i_1}^2 \cdots a_{i_r}^2 < |U(i_1,\ldots i_r)|^2 >_U.$$

Again, by the invariance, $\langle |U(i_1,\ldots i_r)|^2\rangle_U = \langle |U(1,\ldots,r)|^2\rangle_U$, hence make use of the generating function

$$\langle \det(zI-U)(zI-U)^*\rangle_U = \sum_{r=0}^n |z|^{2(n-r)} C_n^r \langle |U(1,\ldots,r)|^2\rangle_U.$$

The evaluation of the left-hand side is a standard random matrix computation which yields $\langle |U(1,\ldots,r)|^2\rangle_U = 1/C_n^r$. Thus finally

$$\begin{aligned} < \det|(zI-AU)|^2 >_U &= \sum_{r=0}^n |z|^{2(n-r)} \frac{e_r(AA^*)}{C_n^r} \\ &= (n+1)\int_0^\infty \frac{\det(I|z|^2 + tAA^*)}{(1+t)^{n+2}}\,dt, \end{aligned}$$

where we have used

$$\frac{1}{C_n^r} = (n+1)\int_0^\infty \frac{t^r\,dt}{(1+t)^{n+2}}$$

For higher moments of $|\det(z - AU)|^2$

- the elementary symmetric functions are replaced by Schur functions (polynomials), and
- the integral representation for $1/C_n^r$ by a Selberg-type integral, (new to the best of my knowledge).

Schur functions

homogeneous symmetric polynomials indexed by partitions $\lambda = (\lambda_1, \ldots \lambda_n)$

$$s_\lambda(x_1, \ldots, x_n) = \det(x_i^{n+\lambda_j - j}) / \det(x_i^{n-j})$$

By convention, $s_\lambda(x) = 0$ if the number of parts of λ is greater than the number of x_j. Also reduction property, $s_\lambda(x_1, \ldots, x_k) = s_\lambda(x_1, \ldots, x_k, 0)$ if the number of parts $\leq k$.

Two examples:

- If $\lambda = (r)$ then $s_\lambda = h_r$, the complete symmetric function of degree r,
- If $\lambda = (1, \ldots, 1)$ then $s_\lambda = e_r$, the elementary symmetric function of degree r.

Have Jacobi-Trudi identities

$$s_\lambda = \det(h_{\lambda_j - j + k}) = \det(e_{\lambda'_j - j + k})$$

where λ' is the partition conjugate to λ (reflection in the diagonal of the Young diagram).

The usefulness of orthogonality

By convention, if M is a matrix with eigvs x_j then $s_\lambda(M) = s_\lambda(x_1, \ldots, x_n)$.

Group theoretic fact: s_λ are irreducible characters of the unitary group, hence orthogonal

$$\int_{U(n)} s_\lambda(U)\overline{s_\mu(U)} d\mu_H(U) = \delta_{\lambda,\mu}$$

If $f(x_1, \ldots, x_n) = \sum_\lambda c_\lambda s_\lambda(x_1, \ldots, x_n)$ then

$$\begin{aligned} c_\lambda &= \int_{U(n)} f(U)\overline{s_\lambda(U)} d\mu_H(U) \\ &\propto \int_{[0,2\pi]^n} f(e^{i\theta_1}, \ldots, e^{i\theta_n}) \det(e^{-i\theta_j(n+\lambda_k-k)}) \det(e^{i\theta_j(n-k)}) d\theta \end{aligned}$$

For multiplicative functions $f = \prod_k g(x_k)$ the above integral can be easily evaluated by the Gram identity

$$c_\lambda = \det(a_{\lambda_k - k + j}), \qquad a_r = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta}) e^{-ir\theta} d\theta$$

Examples

1. Take $\prod_{j,k}(1 - t_k x_j)^{-1} = \prod_j g(x_j)$, where $g(x) = \prod_k (1 - t_k x)^{-1} = \sum_r h_r x^r$. Then $c_\lambda = \det(h_{\lambda_k - k + j})$. Now, by Jacobi-Trudi, $\det(h_{\lambda_k - k + j}) = s_\lambda$, so that

$$\prod_{j,k}(1 - t_k x_j)^{-1} = \sum_\lambda s_\lambda(t) s_\lambda(x) \qquad \text{Cauchy identity.}$$

2. Take $\prod_{j,k}(1 + t_k x_j) = \prod_j g(x_j)$, where $g(x) = \prod_k (1 + t_k x) = \sum_r e_r x^r$. Then $c_\lambda = \det(e_{\lambda_k - k + j})$. Now, by Jacobi-Trudi, $\det(e_{\lambda_k - k + j}) = s_{\lambda'}$, so that

$$\prod_{j,k}(1 + t_k x_j) = \sum_\lambda s_{\lambda'}(t) s_\lambda(x) \qquad \text{Dual Cauchy identity.}$$

3. Take $\exp(x_1 + \ldots + x_n) = \prod_j g(x_j)$, where $g(x) = \exp x = \sum_r x^r / r!$. Then $c_\lambda = \det(1/(\lambda_k - k + j)!)$, so that (e.g. Balantekin, Orlov)

$$e^{\sum_{j=1}^n x_j} = \sum_\lambda c_\lambda s_\lambda(x), \qquad c_\lambda = s_\lambda(1_n) \prod_{j=1}^n \frac{(n-j)!}{(n + \lambda_j - j)!}.$$

Other examples include expansions of $\prod_j (1 \pm x_j)^{\pm\alpha}$, $\alpha > 0$, (Hua identity and its dual).

Examples

Fact: s_λ are also irreducible characters of the general linear group, leading to

$$\int_{U(n)} s_\lambda(AU)\overline{s_\mu(BU)}d\mu_H(U) = \delta_{\lambda,\mu}\frac{s_\lambda(AB^*)}{s_\lambda(I_n)} \tag{1}$$

$$\int_{U(n)} s_\lambda(AUBU^*)d\mu_H(U) = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(I_n)}. \tag{2}$$

4. (due to Balantekin) Expanding $\exp(\mathrm{tr}\, AUBU^*)$ in Schur fncs and then using (2),

$$\int_{U(n)} \exp(\mathrm{tr}\, AUBU^*)d\mu_H(U) = Const.\frac{\det(e^{a_j b_k})}{\Delta(a)\Delta(b)}$$ Itzykson-Zuber

5. (due to Schlittgen & Wettig) Expanding exponentials and then using (1)

$$\int_{U(n)} \exp(\mathrm{tr}(AU + B^*U^*)d\mu_H(U) = Const.\frac{\det(z_j^{k-1}I_{k-1}(z_j))}{\Delta(z^2)}$$

where $z_1^2, \ldots, z_n^2$ are (distinct) eigenvalues of AB^*.

Truncations of unitary matrices and Selberg-type integrals

Setting $A = \mathrm{diag}(1_k, 0_{n-k})$ and $B = \mathrm{diag}(1_m, 0_{n-m})$, $k \le m$, in (2), one cuts the left-top $k \times m$ corner Q of U arriving at [$d\rho$ is the Haar induced measure]

$$\int_{QQ^* \le I_k} s_\lambda(QQ^*) d\rho_{n,k\times m}(Q) = \frac{s_\lambda(1_k)s_\lambda(1_m)}{s_\lambda(1_n)} \tag{3}$$

If $k + m \le n$ then $d\rho_{n,k\times m} \propto \det(I - QQ^*)^{n-k-m} dQ$ (Friedman & Mello '85, and, recently, Neretin, Fyodorov & Sommers, Forrester). Changing vars of integration in (3) yields

$$\frac{1}{Z^n_{k,m}} \int_{[0,1]^k} s_\lambda(q) \prod_{j=1}^{k} q_j^{m-k}(1-q_j)^{n-k-m} \prod_{i<j}(q_i - q_j)^2 dq = \frac{s_\lambda(1_k)s_\lambda(1_m)}{s_\lambda(1_n)}.$$

LHS is a particular case of a Selberg-type integral (general β, Jack p'mials). Conjectured by Macdonald, evaluated by Yan '92, Kaneko '93, Kadell '97.

If $k + m > n$ then $d\rho$ is supported on the boundary of $QQ^* \le I$. The joint p.d.f. of non-trivial eigenvalues of QQ^* is Jacobi-type. Integration formulas available.

Dual Selberg-type integral

Recall the Beta-integral

$$\int_0^1 t^{p-1}(1-t)^{q-1}dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}dt = B(p,q).$$

Correspondingly, the Yan-Kaneko-Kadell generalization of the Selberg integral

$$Const.\int_{QQ^*\le I_k} s_\lambda(QQ^*)\det(I-QQ^*)^{n-k-m}dQ = \frac{s_\lambda(1_k)s_\lambda(1_m)}{s_\lambda(1_n)}, \tag{4}$$

has its dual

$$Const.\int_{QQ^*\ge 0} s_\lambda(QQ^*)\frac{dQ}{\det(I+QQ^*)^{n+k+m}} = \frac{s_\lambda(1_k)s_\lambda(1_m)}{s_{\lambda'}(1_n)} \tag{5}$$

(here Q is $k\times m$). Note the emergence of λ' and no constraints on Q.

The original Selberg integral can be transformed into an integral over $[0,\infty)^n$. However, the integral in (5) requires a separate evaluation (Schur fncs are not preserved by the transformation). The $\beta=1$ integral (involving zonal p'mials) can also be handled. However, $\beta\neq 1,2$ (Jack p'mials) – still to be evaluated.

Angular integrals

n, k, m are positive integers (Think of $n \gg k, m$)

Matrices A, B, U are $n \times n$. Matrices Q are $k \times m$.

By Cauchy identities,

- Selberg-type integral (4) is equivalent to

$$\int\limits_{U(n)} \frac{d\mu_H(U)}{\det(I - AU)^k \det(I - U^* B^*)^m} = \int_{QQ^* \leq I_{\min(k,m)}} \frac{d\rho_{n,k\times m}(Q)}{\det(I - QQ^* \otimes B^* A)},$$

  The above identity holds for $1 \leq k, m \leq n$. Higher moments?

- Dual Selberg-type integral (5) is equivalent to

$$\int_{U(n)} \det(I + AU)^k \det(I + U^* B^*)^m d\mu_H(U) =$$

$$\int_{\mathbb{C}^{k\times m}} \frac{\det(I + QQ^* \otimes B^* A)}{\det(I + QQ^*)^{n+k+m}} dQ.$$

Application to spectral determinants of matrices with complex eigenvalues

Random matrices $W \in \mathbb{C}^{n\times n}$, with invariant ensemble distribution (e.g. $e^{-n\,\mathrm{Tr}\,V(W^*W)}dW$).

Note that joint pdf of eigenvalues is rarely known for such matrices.

In view of unitary invariance,

$$\langle |\det(I + zW)|^{2m}\rangle_W \propto$$

$$\int_{\mathbb{C}^{m\times m}} \langle \det(I + |z|^2 Q^*Q \otimes W^*W)\rangle_W \det(I_m + Q^*Q)^{-n-2m} dQ$$

Thus, integration over angular part of W can be traded for Jacobi average over $m \times m$ matrices Q. Advantage - now have Hermitian matrices W^*W, can apply orthogonal polynomial technique, etc.

Structure - Hankel determinants. Matrix elements are integrals involving orthogonal polynomials.

Colour-flavour transformations (Zirnbauer '96)

- Bosonic version. $\{\vec{x}_j\}_{j=1}^m$ and $\{\vec{y}_j\}_{j=1}^m$ are two sets of vectors in $\mathbb{C}^n$. If $2m \leq n$ then

$$\int_{U(n)} e^{\sum_{j=1}^m (\vec{y}_j^* U \vec{x}_j + \vec{x}_j^* U^* \vec{y}_j)} \, d\mu_H(U) \propto$$

$$\int_{QQ^* \leq I_m} e^{\sum_{j,k=1}^m (Q_{jk} \vec{x}_k^* \vec{x}_j + (Q^*)_{jk} \vec{y}_k^* \vec{y}_j)} \, \det(I - QQ^*)^{n-2m} dQ$$

- Fermionic version. Now $\vec{\chi}_j$, $\vec{\psi}_j$, $\vec{\chi}_j^*$ and $\vec{\psi}_j^*$ are N-component vectors with anti-commuting components. For any m have

$$\int_{U(n)} e^{\sum_{j=1}^m (\vec{\chi}_j^* U \vec{\psi}_j + \vec{\psi}_j^* U^* \vec{\chi}_j)} \, d\mu_H(U) =$$

$$\int_{\mathbb{C}^{m \times m}} e^{\sum_{j,k=1}^m (Q_{jk} \vec{\chi}_k^* \vec{\chi}_j - (Q^*)_{jk} \vec{\psi}_k^* \vec{\psi}_j)} \, \frac{dQ}{\det(I + QQ^*)^{n+2m}}$$

Zirnbauer: algebraic/geometric approach, other classical groups, SUSY variant. In the RMT context useful for evaluating CUE averages of determinantal products.

bCFT and truncations of CUE

X and Y are $n \times m$ with columns $\vec{x}_j$ and $\vec{y}_j$. XY^* has rank m (generically). Hence link to truncations of CUE.

By Schur fnc expansion and Selberg-type integral (4), have

$$\int_{U(n)} e^{\mathrm{Tr}(XY^*U+U^*X^*Y)}\, d\mu_H(U) = \int_{QQ^* \leq I_m} e^{\mathrm{Tr}(QX^*X+Q^*Y^*Y)} d\rho_{n,m\times m}(Q).$$

This is another form of bCFT, but now in the extended range $m \leq n$.

If $2m \leq n$ then $d\rho_{n,m\times m} \propto \det(I - QQ^*)^{n-2m} dQ$ and we are back to Zirnbauer.

CFTs and Selberg type integrals (linked via Schur function expansions):

- bCFT implies (4) and vice versa
- fCFT implies (5). Is the converse true?

Regularised inverse determinants

$$R_\varepsilon(A^*A) = \int_{U(n)} \frac{d\mu_H(U)}{\det[(I+AU)(I+AU)^* + \varepsilon^2 I]^m}$$

Non-trivial even for $m = 1$. Direct application of bCFT runs into a problem (diverging integrals). Schur functions do not help.

A deformed version of CFT: integration over $Q^*Q \leq I_m$ can be replaced by integration over a pair of Hermitian matrices $Q_1 = TPT^*, Q_2 = (T^*)^{-1}PT^{-1}$, where $T \in GL_m(\mathbb{C})$ and $P = \mathrm{diag}(p_1, \ldots, p_m)$, $|p_j| \leq 1$. "Volume element"

$$(dQ_1 dQ_2) = d\mu_H(T) \prod_{j<k} (p_j^2 - p_k^2)^2 \prod_j p_j dp_j$$

Expression for $R_\varepsilon(A^*A)$ in simplest case $m = 1$:

$$\frac{n-1}{2\pi i} \int_0^1 (1-t)^{n-2} dt \int_{-\infty}^{+\infty} \frac{dx}{x} \frac{1}{\det\left[AA^* + (\varepsilon^2 - t)\, I - i\varepsilon\sqrt{t}\left(x + \frac{1}{x}\right) I\right]}.$$