3. Hausdorff Spaces and Compact Spaces

3.1 Hausdorff Spaces

Definition A topological space X is Hausdorff if for any $x, y \in X$ with $x \neq y$ there exist open sets U containing x and V containing y such that $U \bigcap V = \emptyset$.

(3.1a) Proposition *Every metric space is Hausdorff, in particular* $\mathbf{R}^n$ *is Hausdorff (for* $n \geq 1$*).*

Proof Let (X, d) be a metric space and let $x, y \in X$ with $x \neq y$. Let $r = d(x, y)$. Let $U = B(x; r/2)$ and $V = B(y; r/2)$. Then $x \in U$, $y \in V$. We claim $U \bigcap V = \emptyset$. If not there exists $z \in U \bigcap V$. But then $d(x, z) < r/2$ and $d(z, y) < r/2$ so we get

$$r = d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2$$

i.e. $r < r$, a contradiction. Hence $U \bigcap V = \emptyset$ and X is Hausdorff.

Reamrk In a Hausdorff space X the subset $\{x\}$ is closed, for every $x \in X$. To see this let $W = C_X(\{x\})$. For $y \in W$ there exist open set U_y, V_y such that $x \in U_y$, $y \in V_y$ and $U_y \bigcap V_y = \emptyset$. Thus $V_y \subset W$ and $W = \bigcup_{y \in W} V_y$ is open. So $C_X(W) = \{x\}$ is closed.

Exercise 1 Suppose $(X, \mathcal{T})$ is Hausdorff and X is finite. Then $\mathcal{T}$ is the discrete topology.

Proof Let $x \in X$. Then $\{x\}$ is closed. If $Z = \{x_1, x_2, \ldots, x_m\}$ is any subset of X then $Z = \{x_1\} \bigcup \{x_2\} \bigcup \{x_n\}$ is closed. So all subsets are closed and hence all subsets are open and X has the discrete topology.

Exercise 2 Let X be an infinite set and let $\mathcal{T}$ be the cofinite topology on X. Then $(X, \mathcal{T})$ is not Hausdorff.

Lets suppose it is and derive a contradiction. Pick $x, y \in X$ with $x \neq y$. Then there exists open sets U, V such that $x \in U, y \in V$ and $U \bigcap V = \emptyset$. Since X has the cofinite topology and U, V are nonempty, $C_X(U)$ and $C_X(V)$ are finite. But then $X = C_X(\emptyset) = C_X(U \bigcap V) = C_X(U) \bigcup C_X(V)$ is finite - contradiction.

(3.1b) *Let X be a Hausdorff space and let $Z \subset X$. Then Z (regarded as a topological space via the subspace topology) is Hausdorff.*

Proof Let $x, y \in Z$. Since X is Hausdorff there exist open sets U, V in X such that $x \in U$, $y \in V$ and $U \bigcap V = \emptyset$. But then $U^* = U \bigcap Z$ and $V^* = V \bigcap Z$ are open in (the subspace topology on) Z, moreover $x \in U^*$, $y \in V^*$ also

$$U^* \bigcap V^* = U \bigcap Z \bigcap V \bigcap Z = (U \bigcap V) \bigcap Z = \emptyset.$$

Hence Z is Hausdorff.

(3.1c) Proposition *Suppose that X, Y are topological spaces that X is homeomorphic to Y and Y is Hausdorff. Then X is Hausdorff.*

Proof Let $f : X \to Y$ be a homeomorphism. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then $f(x_1), f(x_2) \in Y$ and $f(x_1) \neq f(x_2)$ (as f is a homeomorphism, in particular it is a $1-1$-map). By the Hausdorff condition there exist open sets V_1, V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \bigcap V_2 = \emptyset$. But now $x_1 \in f^{-1}V_1$, $x_2 \in f^{-1}V_2$ and $f^{-1}V_1 \bigcap f^{-1}V_2 = f^{-1}(V_1 \bigcap V_2) = f^{-1}(\emptyset) = \emptyset$. Hence X is Hausdorff.

Definition Suppose $\mathcal{P}$ is a property which a topological space may or may not have (e.g. the property of being Hausdorff). We say that $\mathcal{P}$ is a *topological property* if whenever X, Y are homeomorphic topological spaces and Y has the property $\mathcal{P}$ then X also has the property $\mathcal{P}$.

So we may re-cast (3.1c) as:

(3.1c)' *Hausdorffness is a topological property.*

3.2 Compact Spaces

How can we tell whether $[0,1]$ is homeomorphic to $\mathbf{R}$? Find a topological property which $[0,1]$ has but $\mathbf{R}$ does not have.

Definition Let X be a topological space. An *open cover* of X is a collection of open sets $\{U_i \mid i \in I\}$ such that $X = \bigcup_{i\in I} U_i$.

A *subcover* of an open cover $\{U_i \mid i \in I\}$ is an open cover of the form $\{U_j \mid j \in J\}$, where J is a subset of I.

Examples 1. Let $X = \mathbf{R}$ and let $U_n = (-n, +n)$, for $n = 1, 2, \ldots$. Then $\{U_n \mid n \in \mathbf{N}\}$ is an open cover of $\mathbf{R}$, i.e. $\mathbf{R} = \bigcup_{n=1}^{\infty} U_n$. This is so because, for $r \in \mathbf{R}$ we can choose a positive integer m greater than $|r|$ and then $r \in (-m, +m) = U_m$, so $r \in U_m \subset \bigcup_{n=1}^{\infty} U_n$. Hence $\bigcup_{n=1}^{\infty} U_n$ contains every real number, i.e. $\bigcup_{n=1}^{\infty} U_n = \mathbf{R}$.

A subcover of this open cover is $\{U_n \mid n \in J\}$ where J is the set of even positive integers.

Example 2 Let $X = \mathbf{R}$. Let $U_1 = (-\infty, 0)$, $U_2 = (0, \infty)$, $U_3 = (-1, 1)$, $U_4 = (-4, 4)$, $U_5 = (-5, 5)$ and $U_n = (-n, n)$, for $n \geq 4$. Then $\{U_n \mid n \in \mathbf{N}\}$ is an open cover of $\mathbf{R}$ and $\{U_1, U_2, U_3\}$ is a subcover.

Notice that in both Examples above X is given an open cover consisting of infinitely many sets. In Example 2 there is a finite subcover (a subcover consisting of finitely many sets) and in Example 1 there is not.

Example 3 Let $X = [0,1]$ (with the subspace topology induced from $\mathbf{R}$). Let $U_1 = [0, 1/4)$ and $U_n = (1/n, 1]$, for $n = 2, 3, 4, \ldots$. Then $U_1 = [0,1] \bigcap (-1/4, 1/4)$ is open in the subspace topology and so is $U_n = [0,1] \bigcap (1/n, 2)$, for $n \geq 2$. Note that $\{U_n \mid n = 1, 2, \ldots\}$ is an open cover and $\{U_1, U_5\}$ is a subcover.

Definition A topological space X is *compact* if every open cover of X has a finite subcover, i.e. if whenever $X = \bigcup_{i\in I} U_i$, for a collection of open sets $\{U_i \mid i \in I\}$ then we also have $X = \bigcup_{i\in F} U_i$, for some finite subset F of I.

(3.2a) Proposition *Let X be a finite topological space. Then X is compact.*

Proof Let $X = \{x_1, x_2, \dots, x_n\}$. Let $\{U_i \,|\, i \in I\}$ be an open cover of X. Then $x_1 \in X = \bigcup_{i\in I} U_i$ so that $x_1 \in U_{i_1}$ for some $i_1 \in I$. Similarly, $x_2 \in U_{i_2}$ for some $i_2 \in I$, ..., $x_n \in U_{i_n}$, for some $i_n \in I$.

Let $F = \{i_1, i_2, \dots, i_n\}$. Then $x_r \in U_{i_r} \subset \bigcup_{i\in F} U_i$, for each r. Hence every x in X belongs to $\bigcup_{i\in I} U_i$ and so $X = \bigcup_{i\in F} U_i$, i.e. $\{U_i \,|\, i \in F\}$ is a finite subcover of $\{U_i \,|\, i \in I\}$.

When is a subspace of a topological space compact?

(3.2b) Lemma *Let X be a topological space and let Z be a subspace. Then Z is compact if and only if for every collection $\{U_i \,|\, i \in I\}$ of open sets of X such that $Z \subset \bigcup_{i\in I} U_i$ there is a finite subset F of I such that $Z \subset \bigcup_{i\in F} U_i$.*

Proof ($\Rightarrow$) Suppose Z is compact (regarding Z as a topological space with the subspace topology). Let $\{U_i \,|\, i \in I\}$ be a collection of open sets of X with $Z \subset \bigcup_{i\in I} U_i$. Then we have

$$Z = Z \bigcap Z \subset Z \bigcap (\bigcup_{i\in U} U_i) = \bigcup_{i\in I} Z \bigcap U_i.$$

On the other hand $\bigcup_{i\in I} Z \bigcap U_i \subset Z$ so we have $Z = \bigcup_{i\in I} Z \bigcap U_i$. Writing $V_i = Z \bigcap U_i$ we thus have that all V_i are open in Z (in the subspace topology) and $Z = \bigcup_{i\in I} V_i$. By compactness we therefore have $Z = \bigcup_{i\in F} V_i$ for some finite subset F of I. Now $V_i \subset U_i$ so we get

$$Z = \bigcup_{i\in F} V_i \subset \bigcup_{i\in I} U_i$$

and $Z \subset \bigcup_{i\in F} U_i$, as required.

($\Leftarrow$) Now suppose that Z has the property that whenever $Z \subset \bigcup_{i\in I} U_i$, for open sets U_i in X, there exists a finite subset F of I such that $Z \subset \bigcup_{i\in F} U_i$. We will show that Z is compact. Let $\{V_i \,|\, i \in I\}$ be an open cover of Z. Thus each V_i is open in the subspace topology, so have the form $V_i = Z \bigcap U_i$ for some open set U_i in X. Now we have $Z = \bigcup_{i\in I} V_i \subset \bigcup_{i\in I} U_i$. By the assumed property we therefore have $Z \subset \bigcup_{i\in F} U_i$ for some finite subset F of I. Hence we have $Z = Z \bigcap (\bigcup_{i\in F} U_i) = \bigcup_{i\in F} U_i) = \bigcup_{i\in F} Z \bigcap U_i = \bigcup_{i\in F} V_i$. Thus $\{V_i \,|\, i \in F\}$ is a finite subcover of $\{U_i \,|\, i \in I\}$ and we have shown that every open cover of Z has a finite subcover. Hence Z is compact.

Is a subspace of a compact space compact ? The answer is generally no! We shall see that $[0, 1]$ is compact, but on the other hand $(0, 1)$ is not compact (e.g. $(0, 1) = \bigcup_{n=2}^{\infty} U_n$ where $U_n = (1/n, 1)$ but $\{U_n \,|\, n = 2, 3, \dots\}$ has no finite subcover).

However:

(3.2c) *Let X be a compact topological space and let Z be a closed subset. Then Z is a compact topological space.*

Proof We will use (3.2b) Lemma. So let $\{U_i \mid i \in I\}$ be a collection of open sets in X such that $Z \subset \bigcup_{i\in I} U_i$. Let $I^* = I \bigcup \{\alpha\}$, where α is not in I and set $U_\alpha = C_X(Z)$. Then we claim that $\{U_i \mid i \in I^*\}$ is an open cover of X. Well

$$X = Z \bigcup C_X(Z) \subset \bigcup_{i\in I} U_i \bigcup U_\alpha = \bigcup_{i\in I^*} U_i$$

and certainly $\bigcup_{i\in I^*} U_i \subset X$ so that $X = \bigcup_{i\in I^*} U_i$. But X is compact so that $X = \bigcup_{i\in F^*} U_i$, for some finite subset F^* of I^*.

Now $I^* = I \bigcup \{\alpha\}$ so that

$$F^* = F^* \bigcap I^* = (F^* \bigcap I) \bigcup (F^* \bigcap \{\alpha\}).$$

We set $F = F^* \bigcap I$ so that $F^* = F$ or $F^* = F \bigcup \{\alpha\}$. Thus

$$X = \bigcup_{i\in F^*} = (\bigcup_{i\in F} U_i) \bigcup U_\alpha = (\bigcup_{i\in F} U_i) \bigcup C_X(Z).$$

Hence

$$\begin{aligned} Z = & Z \bigcap X = (\bigcup_{i\in F} Z \bigcap U_i) \bigcup (Z \bigcap C_X(Z)) \\ & = \bigcup_{i\in F} Z \bigcap U_i \end{aligned}$$

(as $Z \bigcap C_X(Z) = \emptyset$). So we have $Z \subset \bigcup_{i\in F} U_i$, for a finite subset F of I. Hence Z is compact, by (3.2b).

What about the converse? If X is a topological space and $Z \subset X$ is such that Z is compact (with respect to the subspace topology) then is Z closed? No! For example take X to be a set with two elements α and β, so $X = \{\alpha, \beta\}$. Regard X as a topological space with the indiscrete topology. Then $Z = \{\alpha\}$ is compact (by (3.2a)) but it is not closed. However:

(3.2d) *Suppose X is a Hausdorff topological space and that $Z \subset X$ is a compact subspace. Then Z is closed.*

Proof We will show that $C_X(Z)$ is open. By (2.2e) it is enough to prove that for each $y \in C_X(Z)$ there exists an open set W_y containing y with $W_y \subset C_X(Z)$.

For each $x \in X$ there are open sets U_x, V_x in X such that $x \in U_x$, $y \in V_x$ and $U_x \bigcap V_x = \emptyset$.

Since $x \in U_x$ for each $x \in Z$ we have

$$Z \subset \bigcup_{x \in Z} U_x.$$

By the compactness of Z and (3.2a) Lemma we have $Z \subset (U_{x_1} \bigcup U_{x_2} \bigcup \cdots \bigcup U_{x_n})$ for a finitely many $x_1, x_2, \ldots, x_n \in Z$. Let $W_y = V_{x_1} \bigcap V_{x_2} \bigcap \cdots \bigcap V_{x_n}$. Then W_y is an open set (since it is the intersection of finitely many open sets) containing y (since each V_x contains y). Suppose $z \in Z \bigcap W_y$. Then $z \in Z \subset U_{x_1} \bigcup U_{x_2} \bigcup U_{x_n}$ so that $z \in U_{x_i}$ for some i. Also $z \in W_y \subset V_{x_i}$ so that $z \in U_{x_i} \bigcap V_{x_i} = \emptyset$, a contradiction. Hence there are no elements in $W_y \bigcap Z$, i.e. $W_y \bigcap Z = \emptyset$ and so $W_y \subset C_X(Z)$.

To summarize : for each $y \in C_X(Z)$ we have produced an open set W_y such that $y \in W_y \subset C_X(Z)$. By (2.2f) Lemma, $C_X(Z)$ is open, i.e. Z is closed.

We now start looking in earnest for compact subsets of $\mathbf{R}^n$. The previous result tells us that any compact $Z \subset \mathbf{R}^n$ must be closed. The next result says that Z cannot be too big.

Definition Let Z be a subset of a metric space X, with metric d. We say that Z is *bounded* if there exists a positive real number N such that $d(z, z') < N$ for all $z, z' \in Z$.

e.g. in $\mathbf{R}$ the subset $\{0, \pm 1, \pm 2, \ldots\}$ is not bounded, but $[0, 1]$ is.

(3.2e) Proposition *Let Z be a subset of a metric space X. If Z is compact (in the subspace topology) then Z is bounded.*

Proof Let $z_0 \in Z$. We claim that

$$X = \bigcup_{n=1}^{\infty} B(z_0; n).$$

By definition each $B(z_0; n) \subset X$ so that RHS $\subset$ LHS. Now let $x \in X$. Then $d(x, z_0) = k$, say, where $k \geq 0$. Pick a positive integer $n > k$. Then we have $x \in B(z_0; n)$ and hence $x \in$ LHS. Hence RHS $\subset$ LHS and so LHS = RHS, i.e. the claim is true.

Now suppose Z is compact. Then $Z \subset \bigcup_{n=1}^{\infty} B(z_0; n)$ and so, by (3.2b) Lemma, we have $Z \bigcup B(z_0; n_1) \bigcup B(z_0; n_2) \bigcup \cdots \bigcup B(z_0; n_r)$, for finitely many open balls $B(z_0; n_1), B(z_0; n_2), \ldots, B(z_0; n_r)$. Let $m = \max\{n_1, n_2, \ldots, n_r\}$. Then we have $Z \subset B(z_0; m)$. Now for $z_1, z_2 \in Z$ we have

$$d(z_1, z_2) \leq d(z_1, z_0) + d(z_0, z_2) \leq m + m = 2m.$$

Hence Z is bounded.

(3.2f) Corollary *Suppose that Z is a compact subset of $\mathbf{R}^n$. Then there exists some $K > 0$ such that for all $t = (t_1, t_2, \ldots, t_n) \in Z$ we have $|t_i| \leq K$, for $1 \leq i \leq n$.*

Proof Regard $\mathbf{R}^n$ as a metric space with the metric

$$d(x, y) = \max\{|x_i - y_i| : i = 1, 2, \ldots, n\}$$

for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ (see Example 3 of Section 2.1). By the above Proposition there exists an N such that $d(x, y) < N$ for all $x, y \in Z$. Fix $s \in Z$. Then form each $t \in Z$ we have

$$d(0, t) \leq d(0, s) + d(s, t) \leq d(0, s) + N.$$

Thus we have $d(0, t) \leq K$, where $K = d(0, s) + N$. For $t = (t_1, \ldots, t_n) \in Z$ we have $d(0, t) = \max\{|t_1|, |t_2|, \ldots, |t_n|\}$ so that $|t_i| \leq K$ for all i and the proof is complete.

(3.2g) $[0, 1]$ *is compact.*

Proof We must show that whenever $[0, 1] \subset \bigcup_{i \in I} U_i$, for a collection of open set $\{U_i \mid i \in I\}$ of $\mathbf{R}$ then there is a finite subset F of I such that $[0, 1] \subset \bigcup_{i \in F} U_i$ (see (3.2b)). We do this by "creeping along" from the left. We let S be the set of all $x \in (0, 1]$ such that there exists some finite subset F_x, say, of I such that $[0, x] \subset \bigcup_{i \in F_x} U_i$. Thus S is the set of x such that $[0, x]$ can be covered by finitely many of the sets U_i.

Step 1 *S is not empty.*
Proof Since $0 \in [0, 1]$ we have $0 \in U_j$ for some $j \in I$ and so $(-r, r) \subset U_j$ for some $r > 0$. Let $s = \min\{r, 1\}$. Then $0 < s/2 < 1$ and $[0, s/2] \subset (-r, r) \subset U_j$. Hence $[0, s/2]$ is covered by finitely many U_i's (one in fact) and so $s/2 \in S$.

Let α be the least upper bound of the set S.

Step 2 $\alpha \in S$.

Proof Note $\alpha \leq 1$. Assume for a contradiction that $\alpha \notin S$. Thus every element of S is less than α. Now $\alpha \in U_j$ for some j so that $(\alpha - r, \alpha + r) \subset U_j$ for some $r > 0$. Since $\alpha - r$ is not an upper bound for S so we have $\alpha - r < \beta < \alpha$ for some $\beta \in S$. There exists a finite subset F, say, of I such that $[0, \beta] \subset \bigcup_{i \in F} U_i$ and moreover we have $[\beta, \alpha] \subset (\alpha - r, \alpha + r) \subset U_j$. Thus we have

$$[0, \alpha] = [0, \beta] \bigcup [\beta, \alpha] \subset \bigcup_{i \in F^*} U_i$$

where $F^* = F \bigcup \{j\}$. But this shows that $\beta \in S$, a contradiction.

Step 3 *Conclusion* $\alpha = 1$.

Proof Suppose not, so that $\alpha < 1$. Now $\alpha \in U_j$ for some $j \in I$ and so $(\alpha - r, \alpha + r) \subset U_j$ for some $r > 0$. Put $s = \min\{r, 1 - \alpha\}$. Then $[\alpha, \alpha + s/2] \subset U_j$ so that $[0, \alpha + s/2] \subset U_j \bigcup \bigcup_{i \in F} U_i$ which implies that $\alpha + s/2 \in S$, contradiction the fact that α is the least upper bound.

Thus $\alpha = 1$ and $[0, 1]$ can be covered by finitely many of the sets U_i. Hence $[0, 1]$ is compact.

(3.2h) *Let X, Y be topological spaces with X compact and let $f : X \to Y$ a continuous map. Then $f(X) = \mathrm{Im}(f)$ is compact.*

Proof Let $Z = \mathrm{Im}(f)$. Let $\{V_i \,|\, i \in I\}$ be a collection of open sets in Y such that $Z \subset \bigcup_{i \in I} V_i$. Then $X = \bigcup_{i \in I} U_i$, where $U_i = f^{-1} V_i$, for $i \in I$. Now X is compact so there is a finite subset F of I such that $X = \bigcup_{i \in F} U_i$. We claim that we have have

$$f(X) \subset \bigcup_{i \in F} V_i.$$

Let $y \in f(X)$ then we can write $y = f(x)$ for some $x \in X$. Since $X = \bigcup_{i \in F} U_i$ we have $x \in U_i = f^{-1} V_i$ for some $i \in F$ and hence $y = f(x) \in V_i$. Thus every $y \in f(X)$ belongs to $\bigcup_{i \in F} V_i$, i.e. $f(X) \subset \bigcup_{i \in I} V_i$. Thus, by (3.2b), $f(X)$ is compact.

(3.2i) Corollary *Compactness is a topological property.*

(3.2j) Proposition *For $a < b$, the closed interval $[a, b]$ is compact.*

Proof Define $f : [0,1] \to [a,b]$ by $f(x) = a + x(b-a)$. Then f is a homeomorphism (with inverse $g : [a,b] \to [0,1]$ given by $g(x) = (x-a)/(b-a)$).

(3.2k) $\mathbf{R}$ *is not homeomorphic to* $[0,1]$.

Proof $[0,1]$ is compact but $\mathbf{R}$ is not (e.g. the open cover $\{U_n \mid n \in \mathbf{N}\}$, with $U_n = (-n,n)$, has no finite subcover).

Remark (3.2l) *If S is a subset of $\mathbf{R}$ which is bounded above then the least upper bound of S belongs to the closure $\overline{S}$. Similarly, if S is bounded below then the greatest lower bound belongs to $\overline{S}$.*
We see this as follows. Let α be the least upper bound. If $\alpha \in S$ then certainly $\alpha \in \overline{S}$. So assume $\alpha \notin S$. For a positive integer n, the number $\alpha - 1/n$ is not an upper bound so there exists $x_n \in S$ with $\alpha - 1/n < x_n \leq \alpha$. Then $\alpha = \lim x_n$ and so lies is $\overline{S}$, by (2.3d), and (2.3b).

Similar remarks apply to the greatest lower bound.

(3.2m) Proposition *Let $f : [a,b] \to \mathbf{R}$ be a continuous function (where $a < b$). Then f is bounded and attains its bounds, i.e. there exists $x_0, x_1 \in [a,b]$ such that $f(x_0) \leq f(x)$ and $f(x) \leq f(x_1)$ for all $x \in [a,b]$.*

Proof Put $Z = \mathrm{Im}(f)$. Then Z is compact, by (3.2h). Hence Z is closed and bounded by (3.2d) and (3.2f). Let β be the least upper bound of Z. Then $\beta \in \overline{Z}$ by (3.2l) and $Z = \overline{Z}$ so that $\beta \in Z$. Hence there exists $x_1 \in [a,b]$ such that $f(x_1) = \beta$. So we have $y \leq f(x_1)$ for all $y \in \mathrm{Im}(f)$, i.e. $f(x) \leq f(x_1)$ for all $x \in [a,b]$.

Similarly there exists $x_0 \in [a,b]$ such that $f(x_0) \leq f(x)$ for all $x \in [a,b]$.

3.3 Product Spaces

Suppose X and Y are topological spaces. We consider the set of points $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$. We would like to regard $X \times Y$ as a topological space. But what should be its open sets? The obvious "try" is to say that a subset of $X \times Y$ is open if (and only if) it has the form $U \times V$, where U is open in X and V is open in Y. Unfortunately this doesn't quite work. The problem is that a union of sets of this form is not generally a set of this form. For example take $X = Y = \mathbf{R}$. Then $(0,1) \times (0,5) \bigcup (0,5) \times (4,5)$ is not a set of the form $U \times V$.

But the remedy is quite straight forward. For topological space X, Y we shall take for the topology all possible unions of sets of the form $U \times V$, with U open in X and V open in Y. There is some checking to be to see that this really works, which we do in (3.3a), (3.3b) below.

Definition Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be topological spaces. Define $\mathcal{V}$ to be the set of subsets of the form $\bigcup_{i \in I} U_i \times V_i$, where I is a set and for each $i \in I$, U_i is an open set in X and V_i is an open set in Y.

(3.3a) *A subset W belongs to $\mathcal{V}$ if and only if for each $w \in W$ there exist open sets $U \subset X$ and $V \subset Y$ with $w \in U \times V \subset W$.*

Proof Suppose that W has the stated property. For each $w \in W$ let U_w be open in X and V_w be open in Y such that $w \in U_w \times V_w \subset W$. Then $\bigcup_{w \in W} U_w \times V_w \subset W$ and $\bigcup_{w \in W} U_w \times V_w$ contains w for each $w \in W$. Hence $W = \bigcup_{w \in W} U_w \times V_w$. Putting $I = W$ we have $W = \bigcup_{i \in I} U_i \times V_i$ so that $W \in \mathcal{V}$.

Conversely suppose $W = \bigcup_{i \in I} U_i \times V_i$. If $w \in W$ then $w \in U_i \times V_i$ for some i so we have $w \in U \times V \subset W$, where $U = U_i$, $V = V_i$.

Remark If X, Y are sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ then $(U_1 \times V_1) \bigcap (U_2 \times V_2) = (U_1 \bigcap U_2) \times (V_1 \bigcap V_2)$.

Proof Exercise.

(3.3b) *$(X \times Y, \mathcal{V})$ is a topological space.*

Proof (i) By (3.3a) we have $\emptyset \in \mathcal{V}$ (there is nothing to check). Also for $W = X \times Y$ and any $w \in W$ taking $U = X, V = Y$ we get $w \in U \times V \subset W$ so that $W = X \times Y$ is in $\mathcal{V}$.
(ii) Suppose that $W_1, W_2 \in \mathcal{V}$. We claim $W_1 \bigcap W_2 \in \mathcal{V}$. Put $W = W_1 \bigcap W_2$. Let $w \in W$. Then by (3.3a) there exist open sets U_1 in X, V_1 in Y such that $w \in U_1 \times V_1 \subset W_1$, and (also by (3.3a)) there exist open sets U_2 in X, V_2 in Y such that $w \in U_2 \times V_2 \subset W_2$. Hence $w \in (U_1 \times V_1) \bigcap (U_2 \times V_2) \subset W_1 \bigcap W_2$, i.e. $w \in (U_1 \bigcap U_2) \times (V_1 \bigcap V_2) \subset W$. Putting $U = U_1 \bigcap U_2$ and $V = V_1 \bigcap V_2$ we have $w \in U \times V \subset W$. Hence $W = W_1 \bigcap W_2$ is in $\mathcal{V}$.
(iii) Now suppose that $\{W_i \mid i \in I\}$ is a collection of sets in $\mathcal{V}$. We claim that $W = \bigcup_{i \in I} W_i$ is in $\mathcal{V}$. Let $w \in W$. Then $w \in W_j$ for some $j \in I$. By (3.3a) there are open sets U in X and V in Y such that $w \in U \times V \subset W_j$. Thus $w \in U \times V \subset W$ (as $W_j \subset W$) so W belongs to $\mathcal{V}$, by (3.3a).

We have now verified the three conditions for $\mathcal{V}$ to be a topology.

Definition We call $\mathcal{V}$ the *product topology* on $X \times Y$ and, as usual, call an element of the topology $\mathcal{V}$ an open set.

Example If X and Y are discrete then $X \times Y$ is discrete.

Example If X and Y are indiscrete then $X \times Y$ is indiscrete.

Example We can now give $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ the product topology. Is this different from the natural topology defined by the metric $d(p,q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, for $p = (x_1, y_1)$, $q = (x_2, y_2)$?

Suppose W is open in the product topology and let $w = (x_0, y_0) \in W$. Then we have $(x_0, y_0) \in U \times V$, for some U, V open. Since U is open and $x_0 \in U$ there exists some $\epsilon_1 > 0$ such that $x \in U$ whenever $|x - x_0| < \epsilon_1$, and, since V is open, there exists some $\epsilon_2 > 0$ such that $y \in V$ whenever $|y - y_0| < \epsilon_2$. Putting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we get $x \in U$, $y \in V$ whenever $|x - x_0| < \epsilon$, $|y - y_0| < \epsilon$. Now if $z = (x, y) \in B_d(w; \epsilon)$ then $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \epsilon$ which implies $|x - x_0| < \epsilon$ and $|y - y_0| < \epsilon$ and so $x \in U, y \in V$. Hence we get $B_d(w; \epsilon) \subset U \times V \subset W$. Thus for any $w \in W$ there exists $\epsilon > 0$ such that $B_d(w; \epsilon) \subset W$. Hence W open in the product topology implies W open in the natural topology on $\mathbf{R}^2$.

Now suppose W is open in the natural topology and let $w = (x_0, y_0) \in W$. Then there exists $\epsilon > 0$ such that $B_d(w; \epsilon) \subset W$. Let $U = \{x \in \mathbf{R} : |x - x_0| < \epsilon/2\}$ and

$V = \{y \in \mathbf{R} : |y - y_0| < \epsilon/2\}$. If $w' = (x', y') \in U \times V$ then

$$d(w, w') = \sqrt{(x - x_0)^2 + (y - y_0)^2} < \sqrt{\epsilon^2/4 + \epsilon^2/4} = \frac{\epsilon}{\sqrt{2}} < \epsilon.$$

Hence $w' \in B_d(w; \epsilon)$. Hence $w' \in U \times V \subset B_d(w; \epsilon) \subset W$. We have shown that for each $w \in W$ there exist open sets U, V in $\mathbf{R}$ such that $w' \in U \times V \subset W$. Hence W is open in the product topology.

We have shown that a subset W of $\mathbf{R}^2$ is open in the product topology if and only it is open in the natural topology on $\mathbf{R}^2$. Hence these topologies coincide.

Example More generally, suppose that (X_1, d_1), (X_2, d_2) are metric spaces. The X_1 and X_2 have natural topological space structures given by the metrics d_1 and d_2.

Let $X = X_1 \times X_2$. Then we can define the "product metric" d on X by $d(z, t) = \max d_1(x_1, y_1), d_2(x_2, y_2)$, for $z = (x_1, x_2)$, $t = (y_1, y_2)$ in $X = X_1 \times X_2$. We claim that the topology given on X by the metric d is actually the product topology.

Let W be open in the product topology and let $w = (x_1, x_2) \in X$. Then we have $w \in U \times V \subset W$ for some open sets U and V of X_1 and X_2. Thus we have $B_{d_1}(x_1; r_1) \subset U$ and $B_{d_2}(x_2; r_2) \subset V$ for some $r_1, r_2 > 0$. Let $s = \min r_1, r_2$. Then we have $B_d(w; s) \subset B_{d_1}(x_1; r_1) \times B_{d_2}(x_2; r_2) \subset U \times V \subset W$. Hence W is open in the metric topology on X.

Conversely suppose W is open in the metric topology on X. Let $w = (x_1, x_2) \in W$. Then we have $B_d(w; s) \subset W$ for some $s > 0$. Moreover, we have $B_{d_1}(x_1; s) \times B_{d_2}(x_2; s) \subset B_d(w; s) \subset W$ and hence $w \in U \times V \in W$, for $U = B_{d_1}(x_1; s)$, $V = B_{d_2}(x_2; s)$. Hence W is open in the product topology.

Example A special case of the above is that the natural topology on $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the product topology and more generally the natural topology on $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$ is the product topology.

There is an amusing characterization of Hausdorff spaces in terms of the product topology.

(3.3c) *Let X be a topological space. Define the diagonal $\Delta = \{(x, x) \,|\, x \in X\}$, a subset of $X \times X$. The space X is Hausdorff if and only if Δ is a closed set in $X \times X$.*

Proof ($\Rightarrow$) Assume X is Hausdorff and prove Δ closed. We have to show
$J = C_{X \times X}(\Delta) = \{(x_1, x_2) \,|\, x_1 \neq x_2\}$ is open.

So let $w = (x_1, x_2) \in J$. Then $x_1 \neq x_2$ so there exist open sets U, V in X such that $x_1 \in U$, $x_2 \in V$ and $U \bigcap V = \emptyset$. We claim that $U \times V \subset J$. If not there exists some element $(t,t) \in U \times V$ (with $t \in X$) but then $t \in U \bigcap V$, and since $U \bigcap V = \emptyset$ this is impossible. So $U \times V \subset J$. We have shown that for each $w \in J$ there exists open sets U, V such that $w \in U \times V \subset J$. Hence J is open (by the definition of the product topology on $X \times X$) and so Δ is closed.
($\Leftarrow$) We assume Δ is closed and prove X is Hausdorff. Thus $J = C_{X \times X}(\Delta)$ is open. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then $(x_1, x_2) \in J$ and J is open so there exist open sets U, V such that $(x_1, x_2) \in U \times V \subset J$. Now $x_1 \in U$, $x_2 \in V$ and $U \bigcap V = \emptyset$ (for if $t \in U \bigcap V$ then $(t,t) \in U \times V \in J$ and J is the set of all elements $(a,b) \in X \times X$ such that $a \neq b$ so this is impossible).

There is another, more important relationship between the product construction and the Hausdorff property.

(3.3d) Proposition *$X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.*

Proof ($\Rightarrow$) We assume $X \times Y$ is Hausdorff and prove X and Y are Hausdorff. Suppose that $x, x' \in X$ with $x \neq x'$. Choose $y \in Y$. Then there are open set W, W' in $X \times Y$ such that $(x, y) \in W$, $(x', y) \in W'$ and $W \bigcap W' = \emptyset$. Since W is open there exist open sets U in X and V in Y such that $(x, y) \in U \times V \subset W$, and similarly there exist open sets U' in X and V' in Y such that $(x', y) \in U' \times V' \subset W'$. We have

$$(U \times V) \bigcap (U' \times V') \subset W \bigcap W' = \emptyset$$

so that $(U \bigcap U') \times (V \bigcap V') = \emptyset$. Now if $t \in U \bigcap U'$ then $(t, y) \in (U \bigcap U') \times (V \bigcap V')$ so there are is no such element t, i.e. $U \bigcap U' = \emptyset$. Thus for $x, x \in X$ with $x \neq x'$ we have produced open sets U, U' such that $x \in U$, $x' \in U'$ and $U \bigcap U' = \emptyset$. Hence X is Hausdorff.

Similarly Y is Hausdorff.

($\Leftarrow$) Now suppose X and Y are Hausdorff. Let $w = (x, y), w' = (x', y') \in X \times Y$ with $w \neq w'$. Then $x \neq x'$ or $y \neq y'$. We assume $x \neq x'$. (The other case is similar.) Then there exist open sets U, U' in X with $x \in U$, $x' \in U'$ and $U \bigcap U' = \emptyset$. Put $W = U \times X$, $W' = U' \times Y$. Then $w \in W, w \in W'$ and $W \bigcap W' = (U \times Y) \bigcap (U' \times Y) = (U \bigcap U') \times Y = \emptyset$. Hence $X \times Y$ is Hausdorff.

Definition Let X, Y be topological spaces. The map $p : X \times Y \to X$, $p(x, y) = x$, is called the *canonical projection onto* X and the map $q : X \times Y \to Y$ is called the *canonical projection onto* Y.

When is a map $f : Z \to X \times Y$ continuous ?

(3.3e) Proposition *Let X, Y be topological spaces.*
(i) The projection maps $p : X \times Y \to X$ and $q : X \times Y \to Y$ are continuous.
(ii) A map $f : Z \to X \times Y$ (where Z is a topological space) is continuous if and only if the maps $p \circ f : Z \to X$ and $q \circ f : Z \to Y$ are continuous.

Proof (i) Let U be an open set in X. Then $p^{-1}U = \{(x, y) \,|\, p(x, y) \in U\} = \{(x, y) \,|\, x \in U\} = U \times Y$ is open in $X \times Y$. Hence p is continuous. Similarly q is continuous.
(ii) $(\Rightarrow)$ If $f : Z \to X \times Y$ is continuous then $p \circ f : Z \to X$ is a composite of continuous maps and hence continuous. Similarly $q \circ f$ is continuous.
$(\Leftarrow)$ Suppose that $p \circ f : Z \to X$ and $q \circ f : Z \to Y$ are continuous. We must prove that f is continuous. So let W be an open set in $X \times Y$. Then we can write $W = \bigcup_{i \in I} U_i \times V_i$, for open sets U_i in X and V_i in Y. We need to show that $f^{-1}W$ is open in Z. We have

$$f^{-1}W = f^{-1}(\bigcup_{i \in I} U_i \times V_i) = \bigcup_{i \in I} f^{-1}(U_i \times V_i)$$

so if each $f^{-1}(U_i \times V_i)$ is open then $f^{-1}W$ will be a union of open sets, hence open. Thus it suffices to prove that for U open in X and V open in Y the set $f^{-1}(U \times V)$ is open in Z.

Now we have

$$f^{-1}(U \times V) = \{z \in Z \,|\, f(z) \in U \times V\}$$
$$= \{z \in Z \,|\, pf(z) \in U \text{ and } qf(z) \in V\}$$
$$= (p \circ f)^{-1}U \bigcap (q \circ f)^{-1}V$$

an intersection of two open sets and hence open.

Example Consider a function $f : \mathbf{R} \to \mathbf{R}^2$, $f(t) = (f_1(t), f_2(t))$ for functions $f_1, f_2 : \mathbf{R} \to \mathbf{R}$. Then $f_1 = p \circ f$, $f_2 = q \circ f$ so f is continuous if and only if both f_1 and f_2 are continuous.

For example $f(t) = (t^2 - t, t^3)$ is continuous since $f_1(t) = t^2 - t$ and $f_2(t) = t^3$ are continuous.

(3.3f) *For topological spaces X, Y the map $\phi : X \times Y \to Y \times X$, $\phi(x, y) = (y, x)$ is a homeomorphism.*

Proof We have the canonical projections $p : X \times Y \to X$, $q : X \times Y \to Y$. We also have the canonical projections $p' : Y \times X \to Y$, $q' : Y \times X \to X$.

Now $p' \circ \phi(x, y) = p'(y, x) = y = q(y)$ so that $p' \circ \phi = q$. In particular $p' \circ \phi$ is continuous. Similarly $q' \circ \phi$ is continuous. Hence ϕ is continuous by (3.3d).

Let $\psi : Y \times X \to X \times Y$ be the map given by $\psi(y, x) = (x, y)$. Then $p \circ \psi = q'$ and $q \circ \psi = p'$ are continuous and hence ψ is continuous, again by (3.3d).

Thus ϕ is a bijection (its inverse is ψ), it is continuous and has continuous inverse (namely ψ). Hence ϕ is a homeomorphism.

We want to show that if X and Y are compact then $X \times Y$ is compact. It is convenient to use the idea of a basis in the proof.

Definition Let X be a topological space with topology $\mathcal{T}$. A *basis* $\mathcal{B}$ is a subset of $\mathcal{T}$ (i.e. $\mathcal{B}$ is a collection of open sets) such that every open set $U \in \mathcal{T}$ is a union of open sets in $\mathcal{B}$, i.e. for any $U \in \mathcal{T}$ there exists an indexing set I and a collection of open sets $\{U_i \,|\, i \in I\}$ in $\mathcal{B}$ such that $U = \bigcup_{i \in I} U_i$.

Example The natural topology on $\mathbf{R}$ has as a basis the set of open intervals, i.e. $\mathcal{B} = \{(a, b) \,|\, a < b\}$ is a basis. In general in a metric space X the sets of the form $B(x; r)$, with $x \in X$ and $r > 0$ form a basis for the topology defined by the metric.

For a discrete space X the one-element sets $\{x\}$ form a basis, i.e. $\mathcal{B} = \{\{x\} \,|\, x \in X\}$ is a basis since, for any U in X we have $U = \bigcup_{x \in U} \{x\}$.

The set of open sets $\mathcal{B} = \{U \times V \,|\, U$ is open in X and V is open in Y$\}$ forms a basis (by the definition of the product topology as the sets of which are unions of sets of this form).

Definition If $(X, \mathcal{T})$ is a topological space and $\mathcal{B} \subset \mathcal{T}$ is a basis then the sets $U \in \mathcal{B}$ are called the *basic open sets.*

(3.3g) Lemma *Let X be a topological space with topology $\mathcal{T}$ and basis $\mathcal{B}$. Then X is compact if and only if every open cover of X by basic open sets has a finite subcover.*

Proof $(\Rightarrow)$ Clear.
$(\Leftarrow)$ Suppose every open cover of X by basic open sets has a finite subcover. We must show that X is compact. So let $X = \bigcup_{i \in I} U_i$ be any open cover. For each $x \in X$ we have $x \in U_i$ for some $i \in I$, call this i_x, so $x \in U_{i_x}$. Now U_{i_x} is a union of basic open sets so there exists some basic open set V_x with $x \in V_x \subset U_{i_x}$. Now $X = \bigcup_{x \in X} V_x$ (since $\bigcup_{x \in X} V_x$ contains all points of X). Hence $\{V_x \,|\, x \in X\}$ is an open cover by basic open sets. Thus, by hypothesis, there exists some finite subset F of X such that $X = \bigcup_{x \in F} V_x$, i.e. $X = V_{x_1} \bigcup \cdots \bigcup V_{x_n}$ where $F = \{x_1, \ldots, x_n\}$. But $V_x \subset U_{i_x}$ so that, putting $j_r = i_{x_r}$, we have $V_{x_r} \subset U_{j_r}$, for $1 \leq r \leq n$. Hence

$$X = V_{x_1} \bigcup \cdots \bigcup V_{x_n} \subset U_{j_1} \bigcup \cdots \bigcup U_{j_n}.$$

Hence

$$X = U_{j_1} \bigcup \cdots \bigcup U_{j_n}$$

and $\{U_{j_1}, \ldots, U_{j_n}\}$ is a finite subcover of $\{U_i \,|\, i \in I\}$. Hence X is compact.

We are now ready for the big one.

(3.3h) Theorem *$X \times Y$ is compact if and only if both X and Y are compact.*

Proof $(\Rightarrow)$ If $X \times Y$ is compact then $X = p(X \times Y)$ and $Y = q(X \times Y)$ are compact by (3.2h).
$(\Leftarrow)$ We assume X and Y are compact. We must show that $X \times Y$ is compact. Let $\mathcal{B}$ be the set of subsets of $X \times Y$ of the form $U \times V$ with U open in X and V open in Y. By (3.3g), to show $X \times Y$ is compact, it suffices to prove that whenever

$$X \times Y = \bigcup_{i \in I} U_i \times V_i \tag{1}$$

then we have $X \times Y = \bigcup_{i \in F} U_i \times V_i$ for some finite subset F of I.

So let's suppose (1) holds. For $x \in X$ we set $I_x = \{i \in I \,|\, x \in U_i\}$. Then

$$Y = \bigcup_{i \in I_x} V_i \qquad (2).$$

The argument for this is as follows. If $y \in Y$ then $(x, y) \in X \times Y$ so $(x, y) \in U_i \times V_i$ for some $i \in I$. Then $x \in U_i$ so $i \in I_x$ and $y \in V_i$ so that $y \in V_i$ for some $i \in I_x$. Hence $y \in \bigcup_{i \in I_x} V_i$ and since y was any element of Y we must have $Y = \bigcup_{i \in I_x} V_i$.

Since Y is compact there is a finite subset F_x of I_x such that

$$Y = \bigcup_{i \in F_x} V_i \qquad (3).$$

We define $U(x) = \bigcap_{i \in F_x} U_i$. Then $U(x)$ is an intersection of finitely many open sets and hence open. Also, $x \in U_i$ so for each $i \in F_x$ so that $x \in U(x)$. Hence we have

$$X = \bigcup_{x \in X} U(x) \qquad (4).$$

Since X is compact we have

$$X = U(x_1) \bigcup U(x_2) \bigcup \cdots \bigcup U(x_n) \qquad (5)$$

for finitely many elements $x_1, x_2, \ldots, x_n$ of X. We now clain that

$$X \times Y = \bigcup_{i \in F} U_i \times V_i \qquad (6).$$

where $F = F_{x_1} \bigcup F_{x_2} \bigcup \cdots \bigcup F_{x_n}$ - a finite set, showing $X \times Y$ to be compact.

So let's prove this claim. Let $(x, y) \in X \times Y$. Then by (5) we have $x \in U(x_r)$ for some $1 \leq r \leq n$. Now by (3) we have $y \in V_{i_0}$, for some $i_0 \in F_{x_r}$, and since $U(x_r) = \bigcap_{i \in F_{x_r}} U_i$, we also have $x \in U_{i_0}$. Hence $(x, y) \in U_{i_0} \times V_{i_0}$ and $i_0 \in F_{x_r} \subset F$ so that $(x, y) \in \bigcup_{i \in F} U_i \times V_i$, as required.

Phew, that was complicated.

Let X, Y be topological space and let $X' \subset X$ and $Y' \subset Y$ be subspaces. Then X' (resp. Y') is a topological space with the induced topology. So we may form the product space $X' \times Y'$. But we may also regard $X' \times Y'$ as a topological space via the subspace topology induced from $X \times Y$. Are these topologies on $X' \times Y'$ the same ? Yes!

(3.3i) Proposition *In the above situation, the product topology on $X' \times Y'$ and the subspace topology on $X' \times Y'$ (given by viewing $X' \times Y'$ as a subspace of the product space $X \times Y$) coincide.*

Proof Let U' be open in X' and V' be open in Y'. Thus we have $U' = U \bigcap X'$ and $V' = V \bigcap Y'$ for some open set U in X and open set V in Y. Thus

$$U' \times V' = (U \bigcap X') \times (V \bigcap Y') = (U \times V) \bigcap (X' \times Y')$$

which is an open set in the subspace topology on $X' \times Y'$ (since $U \times V$ is open in $X \times Y$). Hence any set of the form $U' \times V'$ (with U' open in the subspace topology on X' and V' open in the subsapce topology on Y'). Any open set in the product topology on $X' \times Y'$ is a union of set of this from, hence any subset of $X' \times Y'$ which is open in the product topology on $X' \times Y'$ is a union of sets which are open in the subspace topology and hence open in the subspace topology.

Conversey suppose that W' is open in the subspace topology on $X' \times Y'$. Then we have $W' = (X' \times Y') \bigcap W$, where W is open in $X \times Y$. Now W is a union of sets of the form $U \times V$, with U open in X and V open in Y. Hence W' is a union of sets of the form $(X' \times Y') \bigcap (U \times V)$. However this is $(X' \times U) \bigcap (Y' \bigcap V) = U' \times V'$, where $U' = X' \bigcap U$, $V' = Y' \bigcap V$. Now U' is open in the subspace topology on X' and V' is open in the subspace topology on Y' so that $U' \times V'$ is open in the product topology on $X' \times Y'$. Thus W' is a union of sets which are open in the product topology on $X' \times Y'$ and hence W' is open in the product topology.

We have shown that a subset of $X' \times Y'$ is open in the product topology if and only if it is open in the subspace topology (induced from the product topology on $X \times Y$), as required.

Example Let $a, b, c, d \in \mathbf{R}$ with $a < b$ and $c < d$ and let $Z = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$. This is the subset $[a, b] \times [c, d]$ of $\mathbf{R} \times \mathbf{R}$. The subspace topology on Z, as a subspace of $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$, is the same as the product topology on Z. Note, by (3.2j) and (3.3h), $[a, b] \times [c, d]$ is compact.

(3.3j) *Let X, Y be topological spaces and let $K \subset X$ be closed and let $L \subset Y$ be closed. Then $K \times L$ is a closed subset of $X \times Y$.*

Proof We have

$$C_{X\times Y}(K\times L)=\{(x,y)\in X\times Y\,|\,x\notin K \text{ or } y\notin L\}$$
$$(C_X(K)\times Y)\bigcup(X\times C_Y(L))$$

a union of two open sets and hence open in $X\times Y$. Hence $K\times L$ is closed.

Definition A topological space X is said to be *locally compact* if for each $x\in X$ there exists an open set U and a closed set K with $x\in U\subset K$ and K compact.

Example $\mathbf{R}$ is locally compact. For $x\in\mathbf{R}$ take $U=(x-1,x+1)$ and $K=[x-1,x+1]$.

Exercise Show that local compactness is a topological property.

(3.3k) Proposition *If X,Y are locally compact spaces then $X\times Y$ is locally compact.*

Proof Let $w=(x,y)\in X\times Y$. Since X is locally compact there exists U open K closed and compact with $x\in U\subset K$. Since Y is locally compact there exists V open and L closed and compact with $y\in V\subset L$. Thus we have

$$w=(x,y)\in U\times V\subset K\times L$$

moreover $U\times V$ is open $K\times L$ is closed, by (3.3j), and $K\times L$ is compact, by (3.3h) and (3.3j).

We want to show that $\mathbf{R}^n$ is locally compact, which will imply the Heine-Borel Theorem. But first a general lemma on metric spaces.

(3.3l) Lemma *Let (X,d), (X',d') be metric spaces and let $\phi:X\to X'$ be a map such that $d'(\phi(x_1),\phi(x_2))=d(x_1,x_2)$ for all $x_1,x_2\in X$. Then ϕ is continuous. If ϕ is a bijection then ϕ is a homeomorphism.*

Proof ϕ is continuous by (2.1d). Suppose ϕ is a bijection with inverse ψ. Then we have $d(\psi(x_1'),\psi(x_2'))=d'(\phi(\psi(x_1'),\phi(\psi(x_2'))=d'(x_1',x_2')$ so that ψ is continuous by the first part of the Lemma (with the roles of d and d' reversed). Thus ϕ is a continuous bijection with continuous inverse. Hence ϕ is a homeomorphism.

Exercise Let (X_1,d_1) and (X_2,d_2) be metric spaces. Let $X=X_1\times X_2$ and define $d:X\times X\to\mathbf{R}$ by $d((x_1,x_2),(y_1,y_2))=\max\{d_1(x_1,y_1),d_2(x_2,y_2)\}$, for $(x_1,x_2)\in X$,

$(y_1, y_2) \in X$. Show that d is a metric and that the topology defined by d is the product topology on $X_1 \times X_2$.

(3.3m) Proposition For $n > 1$ the product space $\mathbf{R}^{n-1} \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^n$, regarded as a topological space with the natural topology.

Proof (Note we have already done the case $n = 2$, see the Example following (3.3b).)

An element of $\mathbf{R}^{n-1} \times \mathbf{R}$ is an element (w, z) with $w \in \mathbf{R}^{n-1}$, $z \in \mathbf{R}$. The natural topology on $\mathbf{R}^{n-1}$ is the topology defined by any one of the metrics on $\mathbf{R}^{n-1}$ considered in Section 2.1 (see (2.1c)). The most convenient for our purposes is

$$d_1(w, w') = \max\{|x_1 - x_1'|, \ldots, |x_{n-1} - x_{n-1}'|\}$$

for $w = (x_1, \ldots, x_{n-1})$, $w' = (x_1, \ldots, x_{n-1}')$. The topology on $\mathbf{R}$ is determined by the metric $d_2(x, x') = |x - x'|$. By the above exercise the product topology on $\mathbf{R}^{n-1} \times \mathbf{R}$ is the same as the topology determined by the metric

$$\begin{aligned} d((w, x), (w', x')) &= \max\{d_1(w, w'), d_2(x, x')\} \\ &= \max\{\max\{|x_i - x_i'| : 1 \le i \le n-1\}, |x - x'|\}. \end{aligned}$$

The natural topology on $\mathbf{R}^n$ is defined by the metric d', where

$$d'((x_1, \ldots, x_n), (x_1', \ldots, x_n')) = \max\{|x_i - x_i'| : 1 \le i \le n\}.$$

We define $\phi : \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}^n$ by $\phi((x_1, \ldots, x_{n-1}), x) = (x_1, \ldots, x_{n-1}, x)$. Then, for $w = ((x_1, \ldots, x_{n-1}, x), z = ((y_1, \ldots, y_{n-1}), y) \in \mathbf{R}^{n-1} \times \mathbf{R}$ we have

$$\begin{aligned} d'(\phi(w), \phi(z)) &= d'((x_1, \ldots, x_{n-1}, x), (y_1, \ldots, y_{n-1}, y)) \\ &= \max\{\max\{|x_i - y_i| : 1 \le i \le n-1\}, |x - y|\} \\ &= d(w, z). \end{aligned}$$

Hence $d'(\phi(w), \phi(z)) = d(w, z)$. Hence ϕ is continuous, by (3.3l). Clearly ϕ is a bijection and hence, by (3.3l), ϕ is a homeomorphism.

(3.3n) Proposition $\mathbf{R}^n$ *is locally compact.*

Proof We argue by induction on n. The space $\mathbf{R}$ is locally compact (see the example after the definition of local compactness). Assume now that $n > 1$ and that $\mathbf{R}^{n-1}$ is locally compact. Then $\mathbf{R}^n$ is homeomorphic to $\mathbf{R}^{n-1} \times \mathbf{R}$, by (3.3k). Hence by induction, $\mathbf{R}^n$ is locally compact for all $n \geq 1$.

(3.3o) Proposition *For any $N > 0$ the set $L = \{(x_1, \ldots, x_n) : |x_i| \leq N \text{ for all } i\}$ is a compact space.*

Proof Let d be the metric $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$. This defines the natural topology on $\mathbf{R}^n$. Since $\mathbf{R}^n$ is locally compact there exists an open set U containing $\underline{0} = (0, 0, \ldots, 0)$ and a closed set Z with $\underline{0} \in U \subset Z$ and Z compact. Since U is open we have $B(\underline{0}; r) \subset U$ for some $r > 0$. Let $E = \{y \in \mathbf{R}^n \mid d(\underline{0}; y) \leq r/2\}$. Then E is closed and $E \subset B(\underline{0}; r) \subset U \subset Z$ so that $E \subset Z$. Now Z is compact and E is a closed subset of Z so E is compact, by (3.2c).

We define $\phi : \mathbf{R}^n \to \mathbf{R}^n$ by $\phi((x_1, \ldots, x_n)) = (\frac{N}{r/2} x_1, \ldots, \frac{N}{r/2} x_n)$. This is a continuous map with continuous inverse $\psi((x_1, \ldots, x_n)) = (\frac{r/2}{N} x_1, \ldots, \frac{r/2}{N} x_n)$. So ϕ is a homeomorhism. The restriction of ϕ to E gives a homeomorphism $E \to L$ (with inverse the restriction of ϕ to L). Hence E is homeomorphic to L and since E is compact, L is too, as required.

(3.3p) Heine-Borel Theorem *A subset Z of $\mathbf{R}^n$ is compact if and only if Z is closed and bounded.*

Proof ($\Rightarrow$) Done already, see (3.2d) and (3.2f).
($\Leftarrow$) Suppose Z is closed and bounded. Then Z is a subset of

$$K = \{(x_1, \ldots, x_n) : |x_i| \leq N\}$$

for some $N > 0$. But K is compact by (3.3o) and so Z is a closed subset of a compact space and hence compact, by (3.2c).

(3.3q) Corollary *Let K be a closed, bounded subset of $\mathbf{R}^n$ and let $f : K \to \mathbf{R}$ be a continuous function. Then f is bounded and attains its bounds.*

Proof See the proof of (3.2m).