

Compatible Circuit Decompositions of Eulerian Graphs

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1 Introduction

Let $G = (V, E)$ be an Eulerian graph. Given a bipartition (X, Y) of V , let $E(X, Y)$ denote the set of all edges of G from X to Y . We refer to $E(X, Y)$ as an *edge-cut* of G . A *tour* of G is a closed walk in G which does not repeat edges. A *tour decomposition* of G is a set of tours whose edge-sets partition E . For $e = uv \in E(G)$ we consider e to be made up of two half-edges e_u incident to u and e_v incident to v . (We distinguish between the two half-edges of a loop.) For $v \in V(G)$ let E_v denote the set of half-edges of G incident with v . A *partial transition system* for G is a function T defined on $V(G)$ such that $T(v)$ is either the empty set or else is a partition of E_v into subsets of size two for each $v \in V(G)$. We refer to the vertices of $V(G) - T^{-1}(\emptyset)$ as the *transition vertices* of (G, T) . When v is a transition vertex of G we refer to the elements of $T(v)$ as the *forbidden transitions* at v . We shall say that two partial transition systems for G , T_1 and T_2 , are *compatible* if for each vertex v of G , $T_1(v) \cap T_2(v) = \emptyset$.

A *partial transition system* for G in which each vertex of G is a transition vertex is said to be a *transition system* for G . There is a natural bijection f between transition systems for G and tour decompositions for G : two

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half-edges are consecutive at v in some tour of $f(T)$ if and only if they form a transition in $T(v)$. We use this bijection to extend the concept of compatibility to tour decompositions: a tour decomposition X for G is *compatible* with a system of forbidden transitions T (or another tour decomposition Y) if and only if $f^{-1}(X)$ and T (or $f^{-1}(X)$ and $f^{-1}(Y)$) are compatible systems of forbidden transitions. Given a tour decomposition X for G , we refer to the transitions of $f^{-1}(X)$ as *transitions* of X .

Kotzig [7] showed that if T is a (partial) transition system for an Eulerian graph G , then G has an Euler tour which is compatible to T if and only if each (transition) vertex of G has degree at least four. The analogous problem for compatible circuit decompositions seems to be much more difficult.

Problem 1.1 *Characterize when a partial transition system T for an Eulerian graph G has a compatible circuit decomposition.*

There is a natural link between Problem 1.1 and the Circuit Double Cover Conjecture. Let G be a graph, $2G$ be the graph obtained by replacing each edge of G by two parallel edges, and T be the transition system for $2G$ corresponding to the trail decomposition of $2G$ into the pairs of parallel edges. Then G has a circuit double cover if and only if $(2G, T)$ has a compatible circuit decomposition. Thus one may expect that a solution to Problem 1.1 may lead to a solution of the Circuit Double Cover Conjecture.

An obvious necessary condition for the existence of a compatible circuit decomposition is that (G, T) contains no separating transitions, where a *separating transition* is a pair $\{e_v, f_v\} \in T(v)$, $v \in V$, such that $G - \{e, f\}$ has more components than G . The first named author [2] showed that this condition is also sufficient when G is planar. This result was extended to graphs with no K_5 -minor by Fan and Zhang [1]. We shall discuss examples in Section 2 which show that it is not sufficient in general. The following attractive conjecture made by G. Sabidussi in 1975, see [2], would give a sufficient condition for the existence of a compatible circuit decomposition.

Conjecture 1.2 *If G is an Eulerian graph of minimum degree at least four and T is an Euler tour of G , then G has a circuit decomposition which is compatible to T .*

The purpose of this paper is to formulate a conjectured solution to Problem 1.1 for the case when G is 4-regular, and show that this conjecture can be reduced to its special case when G is essentially 6-edge-connected. We also use a construction due to F. Jaeger to show that our conjecture would

imply the Circuit Double Cover Conjecture. A conjecture equivalent to the one presented here was formulated by the second author in [3] and is shown to imply Conjecture 1.2 in [4]. We refer the reader to [5] for a survey on compatible Euler tours and circuit decompositions and to [6] for a survey on the Circuit Double Cover Conjecture.

2 A recursive construction

Two examples of a 4-regular (and hence Eulerian) graph with a (partial) transition system for which there is no compatible circuit decomposition are:

- *the bad double loop*: this is the graph G with one vertex incident with two loops together with the transition system induced by the decomposition of G into two loops.
- *the bad K_5* : this is the complete graph on five vertices K_5 together with the transition system induced by the decomposition of G into two circuits of length five.

Other examples can be obtained by using the following two operations for combining two graphs with partial transition systems (G_1, T_1) and (G_2, T_2) .

- *1-sum*: $(G, T) = (G_1, T_1) + (G_2, T_2)$. The graph G is obtained from the disjoint union of G_1 and G_2 by deleting an edge $e_i = u_i v_i$ from G_i , $i \in \{1, 2\}$ and then adding two new edges $f = u_1 u_2$ and $h = v_1 v_2$. The partial transition system T is obtained from $T_1 \cup T_2$ by replacing the half-edges $(e_i)_{u_i}$ and $(e_i)_{v_i}$ by the half-edges f_{u_i} and h_{v_i} , respectively, in any transition of T_i which contains them, $i \in \{1, 2\}$.
- *star-product*: $(G, T) = (G_1, T_1) * (G_2, T_2)$. Choose a transition vertex v_1 of degree 4 of (G_1, T_1) and two edges $e_2 = w_2 x_2$ and $f_2 = y_2 z_2$ of G_2 . Let $e_1 = v_1 w_1, f_1 = v_1 x_1, g_1 = v_1 y_1, h_1 = v_1 z_1$ be the edges incident with v_1 where $T_1(v_1) = \{\{(e_1)_{v_1}, (f_1)_{v_1}\}, \{(g_1)_{v_1}, (h_1)_{v_1}\}\}$. The graph G is obtained from the disjoint union of G_1 and G_2 by deleting v_1, e_2, f_2 and then adding four new edges $e = w_1 w_2, f = x_1 x_2, g = y_1 y_2, h = z_1 z_2$. The partial transition system T is obtained from $(T_1 - T_1(v_1)) \cup T_2$ by replacing the deleted half-edges by the new half-edges in any transition which contains them, in an obvious way.

Let $\mathcal{R}$ be the family of pairs (G, T) where G is a 4-regular graph and T is a partial transition system for G which can be obtained recursively from the bad double loop and the bad K_5 as follows.

- If $(G_1, T_1) \in \mathcal{R}$ and (G_2, T_2) is any connected 4-regular graph with a partial transition system then $(G, T) = (G_1, T_1) + (G_2, T_2) \in \mathcal{R}$.
- If $(G_1, T_1), (G_2, T_2) \in \mathcal{R}$ then $(G, T) = (G_1, T_1) * (G_2, T_2) \in \mathcal{R}$.

The following lemmas will imply that no pair $(G, T) \in \mathcal{R}$ can have a compatible circuit decomposition.

Lemma 2.1 *Suppose $(G, T) = (G_1, T_1) + (G_2, T_2)$. Then (G, T) has a compatible circuit decomposition if and only if (G_1, T_1) and (G_2, T_2) both have compatible circuit decompositions.*

Proof: Straightforward. •

Lemma 2.2 *Suppose $(G, T) = (G_1, T_1) * (G_2, T_2)$. If (G, T) has a compatible circuit decomposition then either (G_1, T_1) or (G_2, T_2) has a compatible circuit decomposition.*

Proof: Let $\mathcal{C}$ be a compatible circuit decomposition of (G, T) and let $A = \{e, f, g, h\}$ be the 4-edge-cut of G separating $V(G_1) - v_1$ from $V(G_2)$. We adopt the labelling of edges given in the definition of the star product. Suppose that the edges in A belong to the same circuit C_1 of $\mathcal{C}$. In that case the edges in $E(C_1) \cap E(G_2)$ correspond to two vertex disjoint paths, which determine one or two circuits of G_2 when completed with e_2 and f_2 . Adding the circuits of $\mathcal{C}$ wholly contained in G_2 , we obtain a compatible circuit decomposition of (G_2, T_2) . Hence, we can suppose that A intersects two circuits of $\mathcal{C}$, say C_1 and C_2 . If the circuit of $\mathcal{C}$ containing e also contains f , then the paths determined by $E(C_1) \cap E(G_2)$ and $E(C_2) \cap E(G_2)$ can be extended as before to circuits of G_2 and then to a compatible circuit decomposition of (G_2, T_2) . If not, say $E(C_1) \cap A = \{e, g\}$ and $E(C_2) \cap A = \{f, h\}$, then $(E(C_1) \cap E(G_1)) \cup \{e_1, g_1\}$ and $(E(C_2) \cap E(G_1)) \cup \{f_1, h_1\}$ together with the circuits of $\mathcal{C}$ wholly contained in G_1 form a compatible circuit decomposition of (G_1, T_1) . •

Corollary 2.3 *If $(G, T) \in \mathcal{R}$ then (G, T) has no compatible circuit decomposition.*

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We conjecture that the family $\mathcal{R}$ characterizes the 4-regular graphs G and partial transition systems T for which (G, T) has no compatible circuit decomposition.

Conjecture 2.4 *Let G be a connected 4-regular graph and T be a partial transition system for G . Then (G, T) has no compatible circuit decomposition if and only if $(G, T) \in \mathcal{R}$.*

We say that an edge-cut of a graph G is *trivial* if it consists of all the edges incident with v for some vertex $v \in V(G)$. A graph is *essentially 6-edge-connected* if it is connected and all edge-cuts of size less than six are trivial. It is conjectured in [5, Conjecture 4.3] that every transition system for a 6-edge-connected Eulerian graph G admits a compatible circuit decomposition and stated in the ensuing discussion that this may even be true under the weaker hypothesis that G has minimum degree four and is essentially 6-edge connected (and is not the bad double loop or the bad K_5). The following conjecture is the special case of this statement when G is 4-regular.

Conjecture 2.5 *Let G be an essentially 6-edge-connected 4-regular graph and T be a transition system for G . Then (G, T) has no compatible circuit decomposition if and only if (G, T) is the bad double loop or the bad K_5 .*

Conjecture 2.5 would follow from Conjecture 2.4, since the bad double loop and the bad K_5 are the only essentially 6-edge-connected graphs in $\mathcal{R}$. We shall show in Section 4 that the two conjectures are, in fact, equivalent. We first need to establish some preliminary results on the family $\mathcal{R}$, on compatible circuit decompositions and on the family of 4-edge-connected 4-regular graphs.

Lemma 2.6 *Suppose $(G, T) \in \mathcal{R}$ and $v \in V(G)$ with $T(v) = \emptyset$. Let $T'(v)$ be a partition of E_v into subsets of size two, and put $T' = T \cup T'(v)$. Then $(G, T') \in \mathcal{R}$.*

Proof: We use induction on the number of vertices of G . Since $T(v) = \emptyset$, (G, T) is not the bad double loop or the bad K_5 . Hence we have one of the following two cases.

Case 1 $(G, T) = (G_1, T_1) + (G_2, T_2)$ where $(G_1, T_1) \in \mathcal{R}$ and (G_2, T_2) is arbitrary. For $i \in \{1, 2\}$ let $T'_i = T_i \cup T'(v)$ if $v \in V(G_i)$ and $T'_i = T_i$ otherwise. Then $(G, T') = (G_1, T'_1) + (G_2, T'_2)$. Furthermore, $(G_1, T'_1) \in \mathcal{R}$, by induction if $v \in V(G_1)$, and trivially otherwise. Hence $(G, T') \in \mathcal{R}$.

Case 2 $(G, T) = (G_1, T_1) * (G_2, T_2)$ where $(G_i, T_i) \in \mathcal{R}$ for $i \in \{1, 2\}$. We proceed as in Case 1. •

Lemma 2.7 *Let G be a 4-regular graph, T be a partial transition system for G , and v be a vertex of G which is not a transition vertex of (G, T) . Suppose (G, T) has a compatible circuit decomposition $\mathcal{C}$. Then (G, T) has a compatible circuit decomposition $\mathcal{C}'$ such that either:*

- (a) *The circuits of $\mathcal{C}'$ which contain v intersect only at v , or*
- (b) *The transition systems for G induced by $\mathcal{C}$ and $\mathcal{C}'$ are compatible at v .*

Proof: Let X and Y be the circuits of $\mathcal{C}$ which contain v . Let G' be the graph obtained from $X \cup Y$ by suppressing all vertices of degree two, and X' and Y' be the circuits of G' induced by X and Y , respectively. Let T' be a transition system for G' which agrees with T at each vertex of G' which is a transition vertex of G and uses different transitions to those induced by X' and Y' at each vertex of G' which is transition vertex of G . Let H be the cubic graph obtained by splitting each vertex u of G' into two vertices u_1, u_2 of degree two along the transitions in T' , and then adding a 1-factor $F_1 = \{u_1u_2 : u \in V(G')\}$. Let F_2 and F_3 be the sets of edges of H corresponding to the edges in X' and Y' , respectively. Then $\{F_1, F_2, F_3\}$ is a proper 3-edge-colouring of H .

Let C be the circuit of $F_2 \cup F_3$ which contains v_1 . Let Z be the path or circuit of H defined by putting $Z = C$ if $v_2 \notin V(C)$ and otherwise letting Z be the v_1v_2 -path in C which starts with an edge of F_2 . Let $\{F_1, F'_2, F'_3\}$ be the new 3-edge-colouring of H obtained by interchanging the colours F_2, F_3 along Z . Note that all vertices of H other than perhaps v_1, v_2 are incident with one edge of each colour. Furthermore, if Z is a circuit then v_1, v_2 are also incident with an edge of each colour; if Z is a v_1v_2 -path which enters v_2 along an edge of F_2 then v_1, v_2 are each incident with two edges in F'_3 and one edge in F_1 ; if Z is a v_1v_2 -path which enters v_2 along an edge of F_3 then v_1 is incident with one edge of F_1 and two edges in F'_3 , and v_2 is incident with one edge of F_1 and two edges in F'_2 . In all cases we may obtain a 2-edge-colouring $\{F'_2, F'_3\}$ of G' by contracting F_1 .

Let $\mathcal{D}$ be a tour decomposition of G' in which the edge set of each tour in $\mathcal{D}$ is a monochromatic component in this 2-edge-colouring. By construction, $\mathcal{D}$ is compatible with T' at all vertices of G' other than perhaps v . Furthermore, all tours in $\mathcal{D}$ are circuits with the possible exception that there may be a tour W which contains v as a vertex of degree four and has all other vertices of degree two. In the former case the circuits of $\mathcal{D}$ containing v use different

transitions at v than X' and Y' . In the latter case we may decompose W into two circuits W_1, W_2 with $V(W_1) \cap V(W_2) = \{v\}$. Let $\mathcal{C}_1 = \mathcal{D}$ if $\mathcal{D}$ is a circuit decomposition of G' , and otherwise put $\mathcal{C}_1 = (\mathcal{D} - \{W\}) \cup \{W_1, W_2\}$. Finally let $\mathcal{C}'$ be the compatible circuit decomposition of (G, T) obtained from $\mathcal{C}$ by replacing X and Y by the circuits of G induced by $\mathcal{C}_1$. If $\mathcal{D}$ is a circuit decomposition of G' then $\mathcal{C}$ and $\mathcal{C}'$ are compatible at v . Otherwise the circuits in $\mathcal{C}'$ which contain v are induced by W_1 and W_2 and hence intersect only at v . $\bullet$

Let G be a 4-edge-connected 4-regular graph. Suppose that $E(U_1, U_2) = \{w_1w_2, x_1x_2, y_1y_2, z_1z_2\}$ is a non-trivial 4-edge-cut of G with $\{w_1, x_1, y_1, z_1\} \subseteq U_1$. The *cleavage graphs* of G along $E(U_1, U_2)$ are the graphs G_i , $1 \leq i \leq 2$ obtained by adding a new vertex v and new edges vw_i, vx_i, vy_i, vz_i to $G[U_i]$. It can easily be seen that both cleavage graphs are 4-edge-connected. We call the new vertex v the *marker vertex* of G_i . A partial transition system T for G induces a partial transition system T_i for G_i in an obvious way, taking $T_i(v) = \emptyset$. We say that (G_1, T_1) and (G_2, T_2) are obtained by *cleaving* (G, T) along $E(U_1, U_2)$.

Lemma 2.8 *Suppose that T is a partial transition system for a 4-edge-connected 4-regular graph G , and that (G_1, T_1) and (G_2, T_2) are obtained by cleaving (G, T) along a 4-edge-cut $E(U_1, U_2)$.*

- (a) *For $j \in \{1, 2, 3\}$, let T_i^j be the three distinct transition systems for G_i which agree with T on U_i and for which the marker vertex v is a transition vertex. If (G_i, T_i^j) has a compatible circuit decomposition $\mathcal{C}_i^j$ for all $j \in \{1, 2, 3\}$ and $i \in \{1, 2\}$, then (G, T) has a compatible circuit decomposition.*
- (b) *If (G_1, T_1) has a compatible circuit decomposition $\mathcal{C}_1$ and U_2 contains no transition vertices of T then (G, T) has a compatible circuit decomposition.*

Proof: We adopt the labelling of edges and vertices used in the definition of cleavage graphs.

- (a) We can choose a $j, k \in \{1, 2, 3\}$ such that $\mathcal{C}_1^j$ and $\mathcal{C}_2^k$ ‘agree’ at v . More precisely, we can relabel so that $vw_i, vx_i \in E(C_i)$ and $vy_i, vz_i \in E(D_i)$ for circuits $C_i, D_i \in \mathcal{C}_i^j$ for each $i \in \{1, 2\}$. Putting $C = (C_1 - v_1) \cup (C_2 - v_2)$ and $D = (D_1 - v_1) \cup (D_2 - v_2)$, we have that $\mathcal{C} = (\mathcal{C}_1^j - \{C_1, D_1\}) \cup (\mathcal{C}_2^j - \{C_2, D_2\}) \cup \{C, D\}$ is a compatible circuit decomposition of G .

- (b) This follows in a similar way to (a) since the 4-edge connectedness of G_2 implies that we may choose a circuit decomposition $\mathcal{C}_2$ of G_2 such that $\mathcal{C}_1$ and $\mathcal{C}_2$ ‘agree’ at v . $\bullet$

3 Essentially 6-edge-connected 4-regular graphs

Let $G = (V, E)$ be a graph. For A, B disjoint subsets of V let $d_G(A, B)$ denote the number of edges in G from A to B , and put $d_G(A) = d_G(A, V - A)$. We will suppress the subscript G when it is obvious which graph we are referring to. We need the following well known identities.

Lemma 3.1 *Let $G = (V, E)$ be a graph and $X, Y \subseteq V$. Then:*

- (a) $d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X - Y, Y - X)$;
- (b) $d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, V - (X \cup Y))$.

It can easily be seen that an essentially 6-edge-connected 4-regular graph G of order at least four is simple. Let v be a vertex of G and $e = vw, f = vx, g = vy, h = vz$ be the edges incident to v . The graph $G_v^{e,f}$ obtained by *splitting* v along e, f is obtained from $G - v$ by adding two new edges wx and yz .

Lemma 3.2 *Let $G = (V, E)$ be an essentially 6-edge-connected 4-regular graph with at least six vertices and $v \in V$. Let the edges incident to v be $e_i = vx_i$, for $1 \leq i \leq 4$. Suppose that $G_v^{e_1, e_i}$ is not essentially 6-edge-connected for all $i \in \{2, 3, 4\}$. Then, relabelling x_1, x_2, x_3, x_4 if necessary, we have $\{x_1x_2, x_1x_3, x_1x_4\} \in E$ and $G - \{v, x_1\} \cup \{x_2x_3, x_3x_4, x_4x_2\}$ is essentially 6-edge-connected.*

Proof: Since $G_i := G_v^{e_1, e_i}$ is not essentially 6-edge-connected, there exist sets $X_i \subset V - v$ such that $d_{G_i}(X_i) = 4$, $2 \leq |X_i| \leq |V| - 3$, and $x_1, x_i \in X_i$ for all $i \in \{2, 3, 4\}$. Let $Y_i = (V - v) - X_i$. Since G is essentially 6-edge-connected we must have $\{x_2, x_3, x_4\} - \{x_i\} \subseteq Y_i$.

Consider the sets X_2, X_3 . We have $x_1 \in X_2 \cap X_3$, $x_2 \in X_2 - X_3$, $x_3 \in X_3 - X_2$, and $x_4 \in (V - v) - (X_2 \cup X_3) = Y_2 \cap Y_3$. Applying Lemma 3.1(a) to $G - v$ we have

$$\begin{aligned} 4 + 4 &= d_{G-v}(X_2) + d_{G-v}(X_3) \\ &= d_{G-v}(X_2 \cap X_3) + d_{G-v}(X_2 \cup X_3) + \\ &\quad 2d_{G-v}(X_2 - X_3, X_3 - X_2). \end{aligned} \tag{1}$$

If $|X_2 \cap X_3| \geq 2$ and $|Y_2 \cap Y_3| \geq 2$ then the essential 6-edge-connectivity of G will imply that $d_{G-v}(X_2 \cap X_3) \geq 5$ and $d_{G-v}(X_2 \cup X_3) = d_{G-v}(Y_2 \cap Y_3) \geq 5$. This would contradict (1) and hence either $|X_2 \cap X_3| = 1$ or $|Y_2 \cap Y_3| = 1$.

We may use Lemma 3.1(b) to deduce similarly that either $|X_2 - X_3| = 1$ or $|X_3 - X_2| = 1$. Thus, at least one of the sets X_2, Y_2, X_3, Y_3 has size

two. Similarly, at least one of the sets X_2, Y_2, X_4, Y_4 has size two, and at least one of the sets X_3, Y_3, X_4, Y_4 has size two. Since $|V| \geq 6$, we cannot have $|X_i| = 2 = |Y_i|$ for all $i \in \{2, 3, 4\}$. It follows that we may relabel e_1, e_2, e_3, e_4 if necessary, such that $X_2 = \{x_1, x_2\}$ and $X_3 = \{x_1, x_3\}$. Since $d_{G-v}(X_2) = 4 = d_{G-v}(X_3)$, we have $x_1x_2, x_1x_3 \in E$.

By considering the sets X_3, X_4 , we may use Lemma 3.1(b) and the fact that $x_1x_2 \in E(G-v)$ to deduce that $d_{G-v}(X_4 - X_3) = 3$. The essential 6-edge-connectivity of G now implies that $|X_4 - X_3| = 1$. Thus $X_4 = \{x_1, x_4\}$ and $x_1x_4 \in E$. Hence $\{x_1x_2, x_1x_3, x_1x_4\} \in E$.

To complete the proof of the lemma, we now suppose that $H = G - \{v, x_1\} \cup \{x_2x_3, x_3x_4, x_4x_2\}$ has an edge-cut $E(X, Y)$ of size at most four, with $|X| \geq 2$ and $|Y| \geq 2$. If $\{x_2, x_3, x_4\} \subseteq X$ then $E(X \cup \{v, x_1\}, Y)$ would be an essential edge-cut of G of size at most four. This would contradict the essential 6-edge-connectedness of G . Thus, we may assume by symmetry that $x_2 \in X$ and $x_3, x_4 \in Y$. Then $E(X, Y \cup \{v, x_1\})$ is an essential edge-cut of G of size at most four. This again contradicts the essential 6-edge-connectedness of G . $\bullet$

4 Equivalence of the two conjectures

Theorem 4.1 *Let (G, T) be a counterexample to Conjecture 2.4, chosen such that the number of transition vertices of G is as small as possible and, subject to this condition, such that the number of vertices of G is as small as possible. Then G is essentially 6-edge-connected and T is a transition system for G .*

Proof: Since (G, T) is a counterexample to Conjecture 2.4, (G, T) has no compatible circuit decomposition and $(G, T) \notin \mathcal{R}$.

Claim 1 *G is 4-edge connected.*

Proof: Suppose G is not 4-edge-connected. Then we can express (G, T) as a 1-sum $(G, T) = (G_1, T_1) + (G_2, T_2)$. If both (G_1, T_1) and (G_2, T_2) have compatible circuit decompositions then so would (G, T) . Hence we may assume that (G_1, T_1) has no compatible circuit decomposition. Since (G_1, T_1) has at most as many transition vertices as (G, T) and G_1 has fewer vertices than G , it follows that $(G_1, T_1) \in \mathcal{R}$. Hence $(G, T) \in \mathcal{R}$. This contradicts the fact that G is a counterexample to Conjecture 2.4. Thus G is 4-edge-connected. $\bullet$

Claim 2 G does not have an essential 4-edge-cut which separates two transition vertices of (G, T) .

Proof: We proceed by contradiction. Let $E(U_1, U_2) = \{e, f, g, h\}$ be a non-trivial 4-edge-cut of G such that both U_1 and U_2 contain transition vertices of G . Suppose $e = w_1w_2, f = x_1x_2, g = y_1y_2, h = z_1z_2$ where $w_1, x_1, y_1, z_1 \in U_1$. For $i \in \{1, 2\}$, let (G_i, T_i) be obtained by cleaving G along $E(U_1, U_2)$ and let v be the corresponding marker vertex of G_i . Using Lemma 2.8(a), we may assume without loss of generality that (G_1, T_1^1) has no compatible circuit decomposition, where T_1^1 agrees with T on U_1 and $T_1^1(v) = \{\{vw_1, vx_1\}, \{vy_1, vz_1\}\}$. Since (G_1, T_1^1) has at most as many transition vertices as (G, T) and G_1 has fewer vertices than G , it follows that $(G_1, T_1^1) \in \mathcal{R}$.

Let T' be the partial transition system for G obtained by putting $T'(u) = T(u)$ for all $u \in U_1$, and $T'(u) = \emptyset$ for all $u \in U_2$. Then (G, T') has fewer transition vertices than G . If (G, T') has no compatible circuit decomposition, then the choice of G will imply that $(G, T') \in \mathcal{R}$. This would imply that $(G, T) \in \mathcal{R}$ by Lemma 2.6.

Thus (G, T') has a compatible circuit decomposition, $\mathcal{C}'$. Let $\mathcal{X}'_1$ be the tour decomposition of G_1 obtained from $\mathcal{C}'$ by contracting all edges which join vertices of U_2 . Then $\mathcal{X}'_1$ has at most one element which is not a circuit and, in any such element, v is the unique vertex of degree other than two. Let $\mathcal{C}_1$ be the circuit decomposition of G_1 which is either equal to $\mathcal{X}'_1$ or is obtained from $\mathcal{X}'_1$ by decomposing the unique trail of $\mathcal{X}'_1$ which is not a circuit, into two edge-disjoint circuits. Then $\mathcal{C}_1$ is a compatible circuit decomposition of (G_1, T_1) . Since (G_1, T_1^1) has no compatible circuit decomposition, there are circuits $X_1, Y_1 \in \mathcal{C}_1$ such that $vw_1, vx_1 \in E(X_1)$ and $vy_1, vz_1 \in E(Y_1)$. Using Lemma 2.7 and the fact that every compatible circuit decomposition of (G_1, T_1) must induce the transitions $\{vw_1, vx_1\}$ and $\{vy_1, vz_1\}$ at v , we may modify $\mathcal{C}_1$ if necessary to ensure that $V(X_1) \cap V(Y_1) = \{v\}$.

Let G'_2 be the graph obtained from G_2 by splitting v along vw_2, vx_2 . Let T'_2 be the partial transition system for G'_2 defined by $T'_2(u) = T_2(u)$ for all $u \in U_2$. If (G'_2, T'_2) has a compatible circuit decomposition $\mathcal{C}_2$, then we can combine $\mathcal{C}_1$ and $\mathcal{C}_2$ to obtain a compatible circuit decomposition of (G, T) . Thus (G'_2, T'_2) has no compatible circuit decomposition. Since (G'_2, T'_2) has fewer transition vertices than G , $(G'_2, T'_2) \in \mathcal{R}$. Since $(G, T) = (G_1, T_1^1) * (G'_2, T'_2)$ we have $(G, T) \in \mathcal{R}$. This contradicts the choice of G . •

Claim 3 G is essentially 6-edge-connected.

Proof: Suppose G is not essentially 6-edge-connected. Let W be the set of transition vertices of (G, T) . We may define an equivalence relation on $V(G)$ by saying that two vertices u, v of G are related if every 4-edge-cut which separates u and v in G is trivial. Claim 2 implies that W is contained in an equivalence class U_0 of this relation. Choose an essential 4-edge cut $E(U_1, U'_1)$ of G . Relabelling if necessary, we have $U_0 \subseteq U_1$. Let (G_1, T_1) and (G'_1, T'_1) be obtained by cleaving (G, T) along $E(U_1, U'_1)$, where $U'_1 = V - U_1$. Then W is the set of transition vertices of (G_1, T_1) , and G_1 does not have an essential 4-edge cut which separates two vertices of W . We may continue this process to obtain a sequence $(G_0, T_0), (G_1, T_1), \dots, (G_m, T_m)$ where $(G_0, T_0) = (G, T)$, W is the set of transition vertices of (G_i, T_i) for all $1 \leq i \leq m$, and G_m is essentially 6-edge-connected. Since (G_m, T_m) contains at least one marker vertex v for which $T_m(v) = \emptyset$, (G_m, T_m) is not the bad double loop or the bad K_5 . Thus $(G_m, T_m) \notin \mathcal{R}$. The choice of (G, T) now implies that (G_m, T_m) has a compatible circuit decomposition $\mathcal{C}_m$. We can now apply Lemma 2.8(b) recursively to deduce that (G, T) has a compatible circuit decomposition and hence contradict the choice of G . •

Claim 4 T is a transition system for G .

Proof: Suppose some vertex v is not a transition vertex of (G, T) . Let the edges incident to v be $e_i = vx_i$, for $1 \leq i \leq 4$.

We first consider the case when $|V| \leq 5$. Let $K_n^{(m)}$ be the graph with n vertices in which each pair of vertices is joined by m parallel edges. Since G is 4-regular and essentially 6-edge-connected, G is either the double loop, $K_2^{(4)}$, $K_3^{(2)}$, or K_5 . It can easily be seen that none of these graphs has a partial transition system T with $T(v) = \emptyset$ for some vertex v and for which there is no compatible circuit decomposition. Thus $|V| \geq 6$.

Let $G' = G_v^{e_1, e_i}$, for some $i \in \{2, 3, 4\}$, and T' be the partial transition system for G' induced by T . If (G', T') had a compatible circuit decomposition $\mathcal{C}'$, then $\mathcal{C}'$ would readily give rise to a compatible circuit decomposition for (G, T) . Thus (G', T') has no compatible circuit decomposition. The choice of (G, T) now implies that $(G', T') \in \mathcal{R}$.

Suppose that G' is essentially 6-edge-connected. Since $(G', T') \in \mathcal{R}$, and $|V(G)| \geq 6$, (G', T') is the bad K_5 . We now show directly that (G, T) has a compatible circuit decomposition. Let $f(T')$ be the circuit decomposition of K_5 corresponding to T' . The edges of G' created by the splitting of v

must be independent because G is essentially 6-edge-connected. Depending on whether these edges are part of the same circuit of length 5 in $f(T')$ or not, there are two possible configurations for (G, T) and they both admit compatible circuit decompositions as seen in Figures 1 and 2.

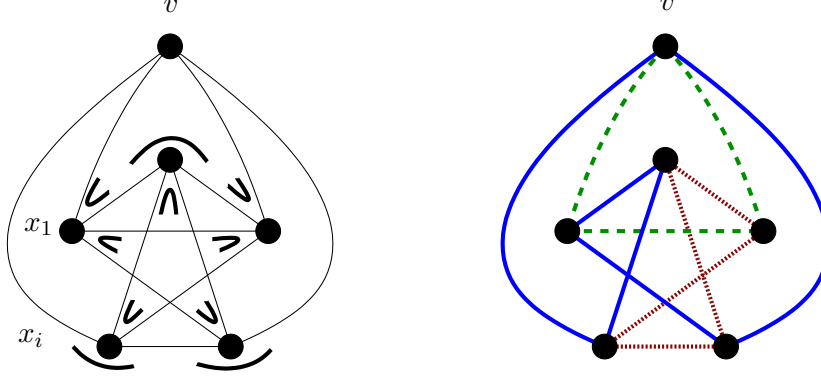


Figure 1: A compatible circuit decomposition for (G, T) in the case when splitting v along e_1, e_i creates two edges which belong to the same 5-circuit in $f(T')$.

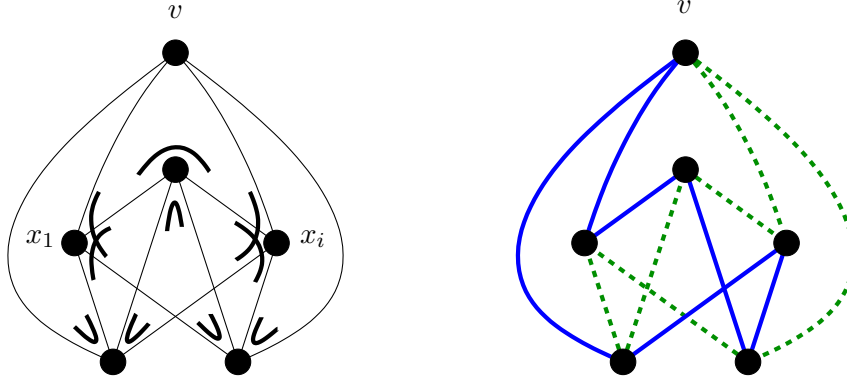


Figure 2: A compatible circuit decomposition for (G, T) in the case when splitting v along e_1, e_i creates two edges which do not belong to the same 5-circuit in $f(T')$.

Thus we may suppose that $G_v^{e_1, e_i}$ is not essentially 6-edge-connected for all $2 \leq i \leq 4$. By Lemma 3.2, there is a relabelling of x_1, x_2, x_3, x_4 such that

$\{x_1x_2, x_1x_3, x_1x_4\} \in E$ and $G'' = G - \{v, x_1\} \cup \{x_2x_3, x_3x_4, x_4x_2\}$ is essentially 6-edge-connected. Let (G_1, T_1) and (G_2, T_2) be obtained by cleaving (G', T') along the 4-edge-cut $E(\{x_1, x_i\}, V(G') - \{x_1, x_i\})$. Then $G_1 = K_3^{(2)}$. Since (G', T') has no compatible circuit decomposition, and every transition system for $K_3^{(2)}$ admits a compatible circuit decomposition, Lemma 2.8(a) implies that (G_2, T_2^j) does not have a compatible circuit decomposition for some transition system T_2^j which agrees with T_2 on $V(G_2) - z$, where z is the marker vertex of G_2 . The choice of (G, T) now implies that $(G_2, T_2^j) \in \mathcal{R}$. Since G_2 is isomorphic to G'' , G_2 is essentially 6-edge-connected, and hence (G_2, T_2^j) must be the bad K_5 . Thus G' is the graph shown on the left of Figure 3 (where we disregard for the moment the indicated transitions of T'). Hence G' is 4-edge-connected and has exactly one non-trivial 4-edge-cut. Since $(G', T') \in \mathcal{R}$, we have $(G', T') = (G_3, T_3) * (G_4, T_4)$ where: (G_3, T_3) is the bad K_5 ; G_4 is the 4-regular graph with two vertices, two loops, and two parallel edges; the star product deletes a vertex from G_3 and the two loops from G_4 . Thus the transitions $T'(x)$ for $x \in V(G') - \{x_1, x_i\}$ are as shown on the left of Figure 3. Hence (G, T) is as shown in the centre of Figure 3. Now let $G^* = G_v^{e_1, e_k}$ for some $k \in \{2, 3, 4\} - \{i\}$, and T^* be the partial transition system for G^* induced by T (see the third graph of Figure 3). By the same reasoning as before, cleaving (G^*, T^*) along its unique non-trivial 4-edge-cut should produce a cleavage graph with a partial transition system that can be extended at the marker vertex to give a bad K_5 . Since this is not possible, we get a contradiction. •

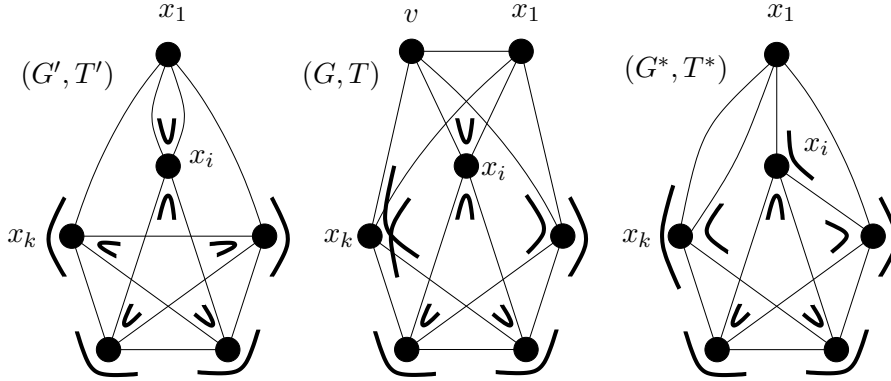


Figure 3: The graphs (G', T') , (G, T) , (G^*, T^*) . The transitions at x_1, x_i , if any, are not shown.

The equivalence of Conjectures 2.4 and 2.5 follows immediately from Theorem 4.1.

We close by using a construction due to Jaeger [6, Section 2.3] to show that the Circuit Double Cover Conjecture would follow from Conjecture 2.5. It is well known that the Circuit Double Cover Conjecture can be reduced to the special case of 3-connected cubic graphs. Let G be such a graph and H be the line graph of G . Then H is 4-regular and essentially 6-edge-connected. There is a natural decomposition of $E(H)$ into triangles Δ_v , $v \in V(G)$, the vertex set of Δ_v being the set of edges of G incident to v . Let T be the transition system of H corresponding to this triangle decomposition. Then (H, T) is not the bad double loop or the bad K_5 so, by Conjecture 2.5, (H, T) has a compatible circuit decomposition $\mathcal{C}$. It is straightforward to check that the vertex sets of the circuits in $\mathcal{C}$ are the edge sets of circuits in a circuit double cover of G .

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