7 Sampling

When all elements of a finite sample space are equally likely calculating probability often boils down to counting the number of ways of making some selection. Specifically, we are often interested in finding how many ways there are of choosing r elements from an n element set. This is called *sampling from an n element set.* The number of such samples depends on exactly what we mean by a sample: is the order important and is repetition allowed.

We assume throughout this section that X is a set with n elements.

Ordered selection with repetition allowed

If we make an ordered selection of r things from X and allow elements to be repeated) then the sample space is

$$S = X^r = \{(x_1, x_2, \ldots, x_r) : x_i \in X \text{ for } 1 \leq i \leq r\},$$

and, since $|X| = n$, we have

$$|S| = n^r.$$

This follows since we have n choices for x_1, for each of these choices there are n choices for x_2, for each of these there are n choices for x_3 and so on. For example, looking back to Exercise Sheet 1 Q1, we are sampling twice from a 6 element set with replacement. We have $S = \{1, 2, 3, 4, 5, 6\}^2$ and $|S| = 36$.

Ordered selection without repetition

If we make an ordered selection of r things from X and we do not allow elements to be repeated then the sample space is

$$S = \{(x_1, x_2, \ldots, x_r) : x_i \in X \text{ for } 1 \leq i \leq r \text{ and } x_i \neq x_j \text{ for } i \neq j\}.$$

To find the cardinality of S notice that there are n choices for x_1, for each of these choices there are $n-1$ choices for x_2, for each of these there are $n-2$ choices for x_3 and so on. Hence,

$$|S| = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!} \tag{1}$$

Note that there are r terms in this product.

We denote the product $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$ by $n!$ and read it as "factorial n." This allows us to rewrite (1) as

$$|S| = \frac{n!}{(n-r)!} \tag{2}$$

By convention we take $0! = 1$.

An important special case of (2) is when $r = n$ i.e. we are making an ordered selection of n things from an n element set. The number of such selections is $n!$ so there are $n!$ different ways to order the elements of an n-element set.

Unordered selection without repetition

If we make an unordered selection of r things from X without repetition then the sample space is just the set of all subsets of X of cardinality r i.e.

$$S = \{Y \subseteq X : |Y| = r\}.$$

An *ordered* selection of r distinct elements of X can be obtained by first choosing an element $Y \in S$ i.e. a subset of X of cardinality r, and then putting its elements in order. Since $|Y| = r$, the elements of Y can be ordered in $r!$ different ways. Thus

$$\begin{aligned}&(\text{number of ordered selections of } r \text{ distinct elements from } X) = \\ &\qquad\qquad |S| \times (\text{number of ways of ordering an } r\text{-element set})\end{aligned}$$

Using the formula for ordered selections without repetition we obtain

$$\begin{aligned}|S| &= \frac{\text{number of ordered selections of } r \text{ distinct elements from } X}{\text{number of ways of ordering an } r\text{-element set}} \\ &= \frac{n!}{(n-r)!\,r!}\end{aligned}$$

The expression $\frac{n!}{(n-r)!\,r!}$ is usually written as $\binom{n}{r}$, and read as "n choose r".

Summary

We have shown the following

Theorem 7.1. *Let X be a set with n elements. Then:*
(a) the number of ways of making an ordered selection of r elements from X with repetitions allowed is n^r.
(b) the number of ways of making an ordered selection of r elements from X with no repetitions allowed is $n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$.
(c) the number of ways of making an unordered selection of r elements from X with no repetitions allowed is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

It is important when answering questions involving sampling that you read the question carefully and decide what sort of sampling is involved. There are many examples in your notes and on exercise sheets. Sometimes more than one kind of sampling can be used but you must be consistent.

The Binomial Theorem

The following is an application of counting selections which will crop up later in the module (and in other modules).

Theorem 7.2 (The Binomial Theorem)**.** *Let n be a nonnegative integer. Then*

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

Proof We have

$$\begin{aligned}(x+y)^n &= \underbrace{(x+y)\times(x+y)\times\cdots\times(x+y)}_{n \text{ factors}} \\ &= a_0 y^n + a_1 x y^{n-1} + a_2 x^2 y^{n-2} + \cdots + a_n x^n\end{aligned}$$

where a_r is the number of ways of making an unordered selection of r elements from an n element set without repetition (we are selecting r factors from the product from which to take an x rather than a y). So $a_r = \binom{n}{r}$ and the result follows. •

Remark The numbers $\binom{n}{r}$ are called *binomial coefficients* because of the role they play in the binomial theorem.