3 Sequences

An ordered n-tuple $(a_1, a_2, \dots, a_n)$ is also called a *sequence* of length n. Sometimes this is written as $(a_i)_{i=1}^n$. When $(a_1, a_2, \dots, a_n)$ is a sequence of numbers, we use the following notation for the *sum* or *product* of terms in the sequence.

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

and

$$\prod_{i=1}^n a_i = a_1 \times a_2 \times \cdots \times a_n.$$

The i here is called a *dummy variable.* Note that

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k$$

so the letter which we choose to represent the dummy variable does not affect the sum (or product).

Proposition 3.1. *Let $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ be sequences of real numbers.*

(a) If $c, d \in \mathbb{R}$ then

$$\sum_{k=1}^n (ca_k + db_k) = c\sum_{k=1}^n a_k + d\sum_{k=1}^n b_k$$

(b)

$$\left(\sum_{i=1}^n a_k\right) \times \left(\sum_{j=1}^n b_k\right) = \sum_{i=1}^n \left(\sum_{j=1}^n a_i b_j\right)$$

The proof is straightforward – just rearrange the terms.

We will also consider infinite sequences. We denote such a sequence by $(a_1, a_2, a_3, \dots)$, or $(a_i)_{i\geq 1}$, or $(a_i)_{i=1}^\infty$. We use the notation

$$\sum_{i=1}^\infty a_i \quad \text{or} \quad \sum_{i\geq 1} a_i$$

for summing infinite sequences of numbers. We can't really say what this means without a little more analysis (Calculus II). One special case we will need for this module, however, is when $a_i = r^i$ for some real number r with $-1 < r < 1$.

Proposition 3.2. *Let r be a real number. Then*

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r} .$$

Furthermore, if $-1 < r < 1$, then

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1 - r} .$$

4 Functions

Definition A *function* (or *map*) from a set X to a set Y is a rule which assigns an element of Y to each element of X. We denote the element of Y assigned to $x \in X$ by $f(x)$, and refer to $f(x)$ as the *value* of f at x. The set X is called the *domain* of f and the set Y is called the *codomain* of f. The *range* of f is the set of all elements of $y \in Y$ for which there exists an $x \in X$ such that $f(x) = y$. In set theoretic notation this is $\{f(x) \ : \ x \in X\}$.

We use the notation

$$f : X \to Y$$

to mean "f is a function from X to Y", and

$$f : x \mapsto f(x)$$

to mean "f maps x to $f(x)$".

Definition Suppose $f : X \to Y$. We say that:

- f is *injective* if for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$ i.e. no two distinct elements of X map to the same element of Y.
- f is *surjective* if for every $y \in Y$ there is an $x \in X$ with $f(x) = y$ i.e. every element in Y has an element of X mapped onto it.
- f is *bijective* if it is both injective and surjective.

An injective function is also called an *injection*. A surjective function is also called a *surjection*. A bijective function is also called a *bijection*.

Lemma 4.1. *Suppose X and Y are finite sets and $f : X \to Y$.*

(a) If f is injective then $|X| \leq |Y|$.

(b) If f is surjective then $|X| \geq |Y|$.

(c) If f is bijective then $|X| = |Y|$.

Proof Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$.
(a) Suppose f is injective. Then $f(x_1), f(x_2), \ldots, f(x_m)$ are all distinct elements of Y. Hence $|Y| \geq m = |X|$.
The proofs of (b) and (c) are left as an exercise. •

Definition Suppose X, Y, Z are sets, $f : X \to Y$ and $g : Y \to Z$. Then the *composite function* $h : X \to Z$ is defined by $h(x) = g(f(x))$. We denote this function h by $g \circ f$.

Note that if $X = Z$ then $g \circ f$ and $f \circ g$ are both functions from X to X but they may be different functions.

Definition Suppose $f : X \to Y$. Then a function $g : Y \to X$ is an *inverse* to f if $(g \circ f)(x) = x$ for all $x \in X$ and $(f \circ g)(y) = y$ for all $y \in Y$.

It is important to remember that not every function has an inverse. The following result characterizes when a function does have an inverse.

Theorem 4.2. *Suppose $f : X \to Y$. Then f has an inverse if and only if f is a bijection.*

Proof There are two things to prove here. Firstly, we must show that if f has an inverse then f is a bijection. Secondly, we must show that if f is a bijection then f has an inverse.

First direction Let $g : Y \to X$ be an inverse for f. Given any $y \in Y$ let $x = g(y)$. Then $f(x) = f(g(y)) = y$. Since this holds for all $y \in Y$, f is surjective. If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$ then applying g to both sides we have $g(f(x_1)) = g(f(x_2))$ so $x_1 = x_2$. Thus, f is injective. Hence f is both surjective and injective, so it is bijective and we have proved the first direction.

Second direction Suppose that f is bijective. Then f is surjective so given any $y \in Y$ there exists an $x \in X$ with $f(x) = y$. Moreover, there is only one such x, since f is injective. Define $g(y)$ to be equal to this x. Then $g(f(x)) = x$ for any $x \in X$ and $f(g(y)) = y$ for any $y \in Y$. So g is an inverse to f. This completes the proof of the second direction. •

Countable sets

Lemma 4.1 suggests a way we can compare the 'cardinalities' of infinite sets.

Definition Let X and Y be infinite sets. We say that

- *the cardinality of* X *is less than or equal to the cardinality of* Y if there exists an injection $f : X \to Y$.
- *the cardinality of* X *is equal to the cardinality of* Y if there exists a bijection $f : X \to Y$.

Recall that $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is the set of *natural numbers*.

Definition let X be an infinite set. We say that X is *countable* if there exists a bijection $f : \mathbb{N} \to X$ i.e. X has the same cardinality as $\mathbb{N}$. If there is no bijection $f : \mathbb{N} \to X$ then we say that X is *uncountable*.

Recall that $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$ is the set of *integers* and $\mathbb{R}$ is the set of *real numbers*. We will show that $\mathbb{Z}$ is countable and $\mathbb{R}$ is uncountable.

Lemma 4.3. $\mathbb{Z}$ *is countable.*

Proof (a) Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} x/2 & \text{if } x \in \mathbb{N} \text{ is even,} \\ -(x+1)/2 & \text{if } x \in \mathbb{N} \text{ is odd.} \end{cases}$$

It can be shown that f is a bijection (see Exercise Sheet 2). Hence $\mathbb{Z}$ is countable. •

Lemma 4.4. $\mathbb{R}$ *is uncountable.*

Proof We use 'proof by contradiction'. (We assume that the statement we are trying to prove is false and show that this assumption implies a contradiction. The only way out of this contradiction is that the original statement must be true.)

Suppose that $\mathbb{R}$ is countable. Then there exists a bijection $f : \mathbb{N} \to \mathbb{R}$. For each $i \in \mathbb{N}$ let $f(i) = r_i$. Since f is surjective, the sequence $(r_0, r_1, r_2, ...)$ must include all of the real numbers. We will obtain our contradiction by constructing a real number r such that $r \neq r_i$ for all $i \geq 0$.

Consider an 'infinite decimal expansion' of each number r_i (if r_i has a finite decimal expansion then we make it infinite by adding an infinite sequence of zeros after the last decimal place). We define a real number r with decimal expansion $r = 0.a_0a_1a_2.....$ as follows. For each $i \in \mathbb{N}$ put $a_i = 5$ if the $(i+1)$'th digit after the decimal point in the decimal expansion of r_i is equal to 0, and otherwise put $a_i = 0$. Then $r \neq r_i$ for all $i \geq 0$ since r and r_i differ in their $(i+1)$'th decimal place. This means that $r \neq f(i)$ for all $i \geq 0$ which contradicts the assumption that f is surjective. The only way out of this contradiction is that $\mathbb{R}$ is uncountable. •