16 Some Common Continuous Probability Distributions

As for the discrete case the probability distributions of some continuous random variables occur so frequently that we give them special names. We look at two such distributions.

16.1 The Uniform Distribution

Suppose that a real number X is chosen from the interval $[a, b]$, in such a way that the probability the number is in any given sub-interval of $[a, b]$ is proportional to the length of the sub-interval. We say that X has the *uniform distribution* on $[a, b]$ and write $X \sim \text{Uniform}[a, b]$ or $X \sim U[a, b]$. Informally, X is equally likely to be anywhere in the interval. It is not difficult to see that the cumulative distribution function and probability density function of X are given by

$$F_X(r) = \begin{cases} 0 & \text{if } r < a \\ \frac{r-a}{b-a} & \text{if } a \leq r \leq b \\ 1 & \text{if } r > b \end{cases}$$

and

$$f_X(r) = \begin{cases} 0 & \text{if } r < a \\ \frac{1}{b-a} & \text{if } a < r < b \\ 0 & \text{if } r > b \end{cases}$$

To find the expectation and variance just substitute this expression for f_X into the definitions and integrate. We obtain $\mathrm{E}(X) = (a + b)/2$ and $\mathrm{Var}(X) = (b - a)^2/12$.

16.2 The Exponential Distribution

The second special distribution we look at is related to the Poisson distribution. Suppose that, on average, λ incidents occur in a unit time interval. Let T be the time at which the first incident occurs. We say that T has the *exponential distribution* and write $T \sim \text{Exponential}(\lambda)$ or $T \sim \text{Exp}(\lambda)$. The cumulative distribution function of T is given by:

$$F_T(r) = \begin{cases} 0 & \text{if } r < 0 \\ 1 - e^{-\lambda r} & \text{if } r \geq 0 \end{cases}$$

To see this choose a fixed $r \geq 0$. On average, $r\lambda$ incidents will occur in a time interval of length r. Thus, if Y is the number of incidents occurring in the time interval $[0, r]$, we will have $Y \sim \text{Poisson}(r\lambda)$. Thus Y is a discrete random variable and $P(Y = k) = e^{-r\lambda}\frac{(r\lambda)^k}{k!}$ for any integer $k \geq 0$. In particular, by taking $k = 0$, we have $P(Y = 0) = e^{-r\lambda}$. Thus

$$\begin{aligned}
F_T(r) &= \mathbb{P}(T \leq r) \\
&= 1 - \mathbb{P}(T > r) \\
&= 1 - \mathbb{P}(\text{there are no incidents in the interval } [0, r]) \\
&= 1 - \mathbb{P}(Y = 0) \\
&= 1 - e^{-\lambda x}.
\end{aligned}$$

Differentiating gives the probability density function

$$f_T(r) = \begin{cases} 0 & \text{if } r < 0 \\ \lambda e^{-\lambda t} & \text{if } r > 0. \end{cases}$$

The expectation and variance of the exponential distribution can be found by integrating (hint: use integration by parts). We obtain:

$$\mathrm{E}(T) = \frac{1}{\lambda}; \quad \mathrm{Var}(T) = \frac{1}{\lambda^2}.$$