Queen Mary
University of London

BSc Examination 2008 By Course Units

MAS108 Probability I

2:30 pm, Wednesday 30 April, 2008
Duration: 2 hours
Do not start reading the question paper until you are instructed to by the invigilators.
The paper has two Sections and you should attempt both Sections.
Please read carefully the instructions given at the beginning of each Section.

Section A: You should attempt all questions. Marks awarded are shown next to the question. This part of the examination carries 60% of the marks.

1. [8 marks]

Let A and B be events with $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 2/3$ and $\mathbb{P}(A \cap B) = 1/3$. Calculate the following probabilities:

a) $\mathbb{P}(B^c)$,

b) $\mathbb{P}(A \cup B)$,

c) $\mathbb{P}(A|B)$,

d) $\mathbb{P}(A \setminus B)$.

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2. [9 marks]

A number is chosen at random from the set $\{1,3,5\}$ with all choices equally likely. Then a number is chosen at random from the set $\{2,4,6\}$ with all choices equally likely.

a) Write down the sample space for this experiment explaining your notation carefully.

b) What is the probability that the first number chosen is larger that the second number chosen?

c) What is the probability that the sum of the two numbers chosen is 7?

d) What is the probability that the two numbers chosen differ by exactly 1?

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3. [8 marks]

A coin which has probability p of coming up heads is tossed three times. Let E be the event "the first toss is a head" and F be the event "exactly two of the three tosses are heads".

a) Show that if the coin is fair (that is $p = 1/2$) then E and F are not independent events.

b) Find a value of p for which E and F are independent.

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4. [8 marks]

Let A be a discrete random variable with probability mass function

n	0	1	2
$P(A=n)$	$5/8$	p	$1/8$

a) Find p.

b) Calculate the following:

i) $\mathbb{E}(A)$,

ii) $\mathrm{Var}(A)$,

iii) $\mathbb{E}(A^3)$.

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5. [9 marks]

a) Write down the probability mass function of a Poisson(λ) random variable.

b) State (without proof) the expectation and variance of a Poisson(λ) random variable.

c) Suppose that $A \sim \text{Poisson}(3)$. Calculate the following probabilities:

i) $\mathbb{P}(A=2)$,

ii) $\mathbb{P}(A \geq 1)$.

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6. [9 marks]

Let U and V be discrete random variables with joint distribution

		U = 0	U = 1	U = 2
	0	$1/12$	$1/12$	$1/12$
V	1	$1/12$	$1/6$	$1/4$
	2	$1/6$	$1/12$	0

a) Find the marginal distributions of U and V.

b) Calculate the covariance of U and V.

7. [9 marks]

Let X be a continuous random variable with cumulative distribution function (cdf) F_X.

a) Define the median of X and the upper and lower quartiles of X.

Suppose that X has cdf

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 3x & \text{if } 0 \le x < \frac{1}{4} \\ \frac{x+2}{3} & \text{if } \frac{1}{4} \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Calculate the following:

b) the median of X,

c) the upper quartile of X,

d) $\mathbb{P}(\frac{1}{12} \le X \le \frac{1}{2})$.

Section B: Each question carries 20 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted. This part of the examination carries 40% of the marks.

8.

Let A and B be events with $\mathbb{P}(A), \mathbb{P}(B) > 0$.

a) State and prove Bayes' theorem.

b) Prove that if $\mathbb{P}(A|B) > \mathbb{P}(A)$ then $\mathbb{P}(B|A) > \mathbb{P}(B)$.

c) Prove that if $\mathbb{P}(A|B) > \mathbb{P}(A)$ then $\mathbb{P}(A \cap B) > \mathbb{P}(A)\mathbb{P}(B)$.

The weather forecast for tomorrow reports that there is a 30% chance of rain, a 20% chance of snow and otherwise it will be fine. Assume that these are accurate probabilities and that the weather will be exactly one of rainy, snowy or fine.
The probability that my train to work is late is $1/2$ on a fine day, $3/4$ on a rainy day and 1 on a snowy day.

d) What is the probability that my train to work tomorrow is late?

e) Suppose that the train is late. What is the conditional probability that it rained?

f) Use your answer to part e) to illustrate the result you proved in part b).

9.

Let n and r be integers with $0 \le r \le n$.

a) Write down a formula for the binomial coefficient $\binom{n}{r}$.

b) Explain in detail how the binomial coefficient $\binom{n}{r}$ arises as the number of ways of making a certain selection (sample) of objects.

A bag contains 12 marbles, 7 of which are red and 5 of which are blue. I reach into the bag and take out 3 marbles at random (with all choices equally likely).

c) What is the probability that the 3 selected marbles are all red?

d) What is the probability that the 3 selected marbles are all the same colour?

e) Given that the 3 selected marbles are all the same colour what is the conditional probability that they are all red?

I now pick a fourth marble at random from the bag without replacing the 3 selected marbles.

f) Given that the first 3 marbles are all the same colour what is the conditional probability that the fourth also has the same colour?

10.

a) Define the cumulative distribution function (cdf) of a random variable X.

b) In the case that X is a continuous random variable say how to find the probability density function (pdf) of X given the cdf.

Let S be a disc of radius 3. Let d be a point chosen randomly from within the disc with the probability that d is in any fixed region being proportional to the area of the region. Let D be the random variable "the distance from the centre of the disc to the point d".

c) Find the cdf of D.

d) Find the pdf of D.

e) Find $\mathbb{E}(D)$.

A new random variable C is defined by $C = \frac{1}{7D}$. Let F_C be the cdf of C.

f) Show that if $x > 0$ then

$$F_C(x) = 1 - F_D\left(\frac{1}{7x}\right).$$

11.

a) Explain carefully how a $\text{Bin}(n, p)$ distribution arises in the context of a sequence of independent Bernoulli trials.

A football team plays 20 matches in a season. Each match may be won, lost, drawn or abandoned. Suppose that for each match the probability of winning is $2/5$, the probability of losing is $3/10$, the probability of the match being drawn is $1/5$ and the probability of the match being abandoned is $1/10$. Suppose also that the outcome of each match is independent of the outcomes of all other matches.

b) What is the probability that the team do not win any of the 20 matches in the season?

c) Let W be the number of matches won in the season. Explain why the random variable W has a binomial distribution and give the parameters (the n and the p) of the distribution.

d) What is the expectation of W?

e) What is the distribution of the number of matches which do not finish decisively (that is they are either drawn or abandoned).

f) Suppose that the team scores 3 points for a win, 0 points for a loss and 1 point for a draw or abandoned match. Let P be the number of points acccumulated during the season. What is the expectation of P?

END OF EXAM