MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 9

Q1. (a) Each of A and B is a random variable taking values in $\{0, 1, 2\}$. To find the joint distribution we need to find the probability

$$\mathbb{P}(A = i \text{ and } B = j)$$

for each pair $i, j \in \{0, 1, 2\}$.

$$\mathbb{P}(A = 1, B = 0) = \mathbb{P}(\{(1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (4, 1), (5, 1), (6, 1)\}) = 8/36 = 2/9,$$

where, as usual we are expressing the outcome "1st die shows i, 2nd die shows j" by (i, j).)
Working this out similarly for all pairs we get the joint distribution:

		A		
		0	1	2
	0	4/9	2/9	1/36
B	1	2/9	1/18	0
	2	1/36	0	0

(b) A and B are not independent since, for example,

$$\mathbb{P}(A = 2, B = 2) = 0 \neq \frac{1}{36} \times \frac{1}{36} = \mathbb{P}(A = 2)\mathbb{P}(B = 2)$$

(c) To find the covariance it is probably simplest to use the formula from Proposition 13.6(a) in the notes. That is

$$\mathrm{Cov}(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B).$$

The only outcome with positive probability for which AB is non-zero is $A = 1, B = 1$; this outcome has probability $1/18$ and so $\mathbb{E}(AB) = 1 \times 1 \times 1/18 = 1/18$. You can work out $\mathbb{E}(A)$ by finding the probability mass function of A (given by the column sums in the above table).

x	0	1	2
$P(A = x)$	25/36	5/18	1/36

and so $\mathbb{E}(A) = 1 \times 5/18 + 2 \times 1/36 = 1/3$. Similarly $\mathbb{E}(B) = 1/3$. It follows that,

$$\mathrm{Cov}(A, B) = 1/18 - 1/3 \times 1/3 = -1/18.$$

The correlation coefficient is defined to be

$$\frac{\mathrm{Cov}(A,B)}{\sqrt{\mathrm{Var}(A)\mathrm{Var}(B)}}.$$

In our case, we can find $\mathrm{Var}(A) = 5/18$, $\mathrm{Var}(B) = 5/18$ (using the marginal distributions) and so

$$\mathrm{corr}(A,B) = \frac{-1/18}{\sqrt{5/18 \times 5/18}} = -1/5.$$

Q2. Firstly, you can work out the expectation and variance of each of X, Y and Z (this is an exercise in what you remember about distributions).

$$\begin{aligned} \mathbb{E}(X) &= 7/6 \quad \mathrm{Var}(X) = 35/36 \\ \mathbb{E}(Y) &= 2 \quad \mathrm{Var}(Y) = 2 \\ \mathbb{E}(Z) &= 6 \quad \mathrm{Var}(Z) = 6 \end{aligned}$$

(i) $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 7/6 + 2 = 19/6$.

(ii) $\mathbb{E}(X+Z) = \mathbb{E}(X) + \mathbb{E}(Z) = 7/6 + 6 = 43/6$ (It doesn't matter that X and Z are not independent the expectation of their sum is still the sum of their expectations).

(iii) $\mathbb{E}(X+2Y+3Z) = \mathbb{E}(X) + 2\mathbb{E}(Y) + 3\mathbb{E}(Z) = 7/6 + 4 + 18 = 139/6$ (Again it doesn't matter that we don't have independence.......)

(iv) ... but here we do need independence). Because X and Y are independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 35/36 + 2 = 107/36$.

(v) $\mathrm{Var}(X+Z)$ cannot be determined.

(vi) $\mathrm{Var}(X+2Y+3Z)$ cannot be determined.

Q3 (a) We use induction on n.
Base Case Suppose $n = 2$. We have

$$\mathbb{E}(c_1X_1 + c_2X_2) = \mathbb{E}(c_1X_1) + \mathbb{E}(c_2X_2) = c_1\mathbb{E}(X_1) + c_2\mathbb{E}(X_2)$$

by Propositions 13.2 and 11.2(a).
Induction Hypothesis Suppose that $n \geq 3$ and that, for all integers m with $2 \leq m < n$, we have

$$\mathbb{E}\left(c_1X_1 + c_2X_2 + \ldots + c_mX_m\right) = c_1\mathbb{E}(X_1) + c_2\mathbb{E}(X_2) + \ldots + c_m\mathbb{E}(X_m)$$

Inductive Step Let $X = c_1X_1 + c_2X_2 + \ldots + c_{n-1}X_{n-1}$. Then

$$\mathbb{E}(X + c_nX_n) = \mathbb{E}(X) + c_n\mathbb{E}(X_n)$$

by the base case. We also have

$$\mathbb{E}(X) = c_1\mathbb{E}(X_1) + c_2\mathbb{E}(X_2) + \ldots + c_{n-1}\mathbb{E}(X_{n-1})$$

by the induction hypothesis (with $m = n - 1$). Thus

$$\mathbb{E}\left(c_1X_1 + c_2X_2 + \ldots + c_nX_n\right) = c_1\mathbb{E}(X_1) + c_2\mathbb{E}(X_2) + \ldots + c_n\mathbb{E}(X_n)$$

and the statement is also true for n random variables.
(b) We use induction on n.
Base Case Suppose $n = 2$. We have

$$\mathrm{Var}(c_1X_1 + c_2X_2) = \mathrm{Var}(c_1X_1) + \mathrm{Var}(c_2X_2) = c_1^2\mathrm{Var}(X_1) + c_2^2\mathrm{Var}(X_2)$$

by Propositions 13.4 and 11.3(b).
Induction Hypothesis Suppose that $n \geq 3$ and that, for all integers m with $2 \leq m < n$, we have

$$\mathrm{Var}\left(c_1X_1 + c_2X_2 + \ldots + c_mX_m\right) = c_1^2\mathrm{Var}(X_1) + c_2^2\mathrm{Var}(X_2) + \ldots + c_m^2\mathrm{Var}(X_m)$$

Inductive Step Let $X = c_1X_1 + c_2X_2 + \ldots + c_{n-1}X_{n-1}$. Then X and X_n are independent and

$$\mathrm{Var}(X + c_nX_n) = \mathrm{Var}(X) + c_n^2\mathrm{Var}\mathbb{E}(X_n)$$

by the base case. We also have

$$\mathrm{Var}(X) = c_1^2\mathrm{Var}(X_1) + c_2^2\mathrm{Var}(X_2) + \ldots + c_{n-1}^2\mathrm{Var}(X_{n-1})$$

by the induction hypothesis (with $m = n - 1$). Thus

$$\mathrm{Var}\left(c_1X_1 + c_2X_2 + \ldots + c_nX_n\right) = c_1^2\mathrm{Var}(X_1) + c_2^2\mathrm{Var}(X_2) + \ldots + c_n^2\mathrm{Var}(X_n)$$

and the statement is also true for n independent random variables.

Please let me know if you have any comments or corrections