MTH4107 Introduction to Probability – 2010/11
Exercise Sheet 8

These questions are on discrete random variables. You should write up your solution to the starred question, Q5, **clearly** *and hand it in during your week 11 exercise class for feedback. Put your* **full name and student number** *on the top of your solution. It is important that you make a serious attempt to do* **all** *of questions Q1-Q5 before week 11 lectures begin. Questions AQ1-AQ3 are for additional practice. You should attempt them when you have time.*
In addition to your lecture notes material relating to these questions can be found in Devore Chapter 3 sections 3.4-6, or Ross chapter 4 sections 4.6-8.

Q1. Let X be a discrete random variable with $\mathbb{E}(X) = 5$, $\text{Var}(X) = 2/3$. Find the following:

(a) $\mathbb{E}(3X)$; (b) $\text{Var}(3X)$; (c) $\mathbb{E}(4 - 3X)$; (d) $\text{Var}(4 - 3X)$; (e) $\mathbb{E}(4 - 3X^2)$.

Q2. A standard fair die is rolled repeatedly. For each of the following distributions write down in words a random variable related to this situation which has the stated distribution. In each case give the expectation and variance.

(a) $\text{Bin}(6, 1/6)$; (b) $\text{Bin}(8, 1/2)$; (c) $\text{Geom}(1/6)$; (d) $\text{Geom}(2/3)$.

Q3. Suppose that $A \sim \text{Geom}(1/3)$ and that $B \sim \text{Poisson}(3)$. Find the following probabilities. (You can leave your answers involving powers of e but you should simplify all factorials and other powers.)

(a) $\mathbb{P}(A = 3)$; (b) $\mathbb{P}(A \leq 3)$; (c) $\mathbb{P}(B = 2)$; (d) $\mathbb{P}(B > 2)$.

Q4 Suppose that $X \sim \text{Geom}(p)$. Prove that $\mathbb{E}(X^2) = \frac{2(1-p)}{p^2} + \frac{1}{p}$. Deduce that $\text{Var}(X) = (1 - p)/p^2$. (You can use similar techniques to those we used to determine $\mathbb{E}(X)$ in the lecture notes.)

Q5*. Prove the following statements about a discrete random variable X. (You can use similar techniques to those we used to determine $\mathbb{E}(X)$ in the lecture notes. You also need to use the fact that $\mathbb{E}(X[X - 1]) = \mathbb{E}(X^2) - \mathbb{E}(X)$ which we will prove in Section 13.)
(a) Suppose that $X \sim \text{Bin}(n, p)$. Prove that $\mathbb{E}(X[X - 1]) = n(n - 1)p^2$. Deduce that $\text{Var}(X) = np(1 - p)$.
(b) Suppose that $X \sim \text{Poisson}(\lambda)$. Prove that $\mathbb{E}(X[X - 1]) = \lambda^2$. Deduce that $\text{Var}(X) = \lambda$.

AQ1. A fair coin is tossed four times. Let N be the number of instances of a head followed by another head in the sequence of tosses.
A student argues as follows: There are three possible ways in which we could have a head followed by another head (at the first and second, the second and third, or third and fourth toss). We have a probability $1/2 \times 1/2 = 1/4$ of getting a head followed by another head at each of these positions. Hence N is the number of successes in three Bernoulli(1/4) trials and so $N \sim \text{Bin}(3, 1/4)$.

Explain what is wrong with this argument and determine the distribution of N.

How do the expectation and variance of N differ from the expectation and variance of a $\text{Bin}(3, 1/4)$ random variable? Comment on what this means.

AQ2. Let G be a $\text{Geom}(p)$ random variable.

(a) Show that for any $k, l \geq 1$

$$\mathbb{P}(G > k + l \,|\, G > k) = \mathbb{P}(G > l).$$

(b) Why do you think this is sometimes called the "memoryless property" of the geometric distribution?

AQ3. Let X be the number of fish caught by a fisherman in one afternoon. Suppose that X is distributed $\text{Poisson}(\lambda)$. Each fish has probability p of being a salmon independently of all other fish caught. Let Y be the number of salmon caught. Show that $Y \sim \text{Poisson}(p\lambda)$.

Hint: Use the Theorem of total probability to first show that

$$\mathbb{P}(Y = m) = \sum_{i \geq m} \mathbb{P}(Y = m \,|\, X = i)\mathbb{P}(X = i).$$