MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 7

Q1.

(a) (i) $\mathbb{P}(X = 2) = 1/20$ (just read it off the table)

(ii) $\mathbb{P}(X = 3) = 0$ (since 3 is not a value that X takes)

(iii) $\mathbb{P}(X \leq 1) = \mathbb{P}(X = -2) + \mathbb{P}(X = -1) + \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = 19/20$

(iv) $\mathbb{P}(X < 1) = \mathbb{P}(X = -2) + \mathbb{P}(X = -1) + \mathbb{P}(X = 0) = 3/4$

(v) $\mathbb{P}(X^2 < 1) = \mathbb{P}(X = 0) = 1/4$

(b)

$$\mathrm{E}(X) = (-2)\times(1/10)+(-1)\times(2/5)+(0)\times(1/4)+(1)\times(1/5)+(2)\times(1/20) = -3/10.$$

We have $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$. Since

$$\mathrm{E}(X^2) = (4)\times(1/10)+(1)\times(2/5)+(0)\times(1/4)+(1)\times(1/5)+(4)\times(1/20) = 6/5$$

this gives $\mathrm{Var}(X) = 6/5 - (-3/10)^2 = 6/5 - (9/100) = 111/100$.

(c) We have $-2^2+4 = 8$, $-1^2+4 = 5$, $0^2+4 = 4$, $1^2+4 = 5$ and $2^2+4 = 8$. Hence $\mathit{Range}(Y) = \{4, 5, 8\}$.
The probability mass function of Y is given by

$$\begin{aligned}\mathbb{P}(Y = 4) &= \mathbb{P}(X = 0) = 1/4,\\ \mathbb{P}(Y = 5) &= \mathbb{P}(X = -1) + \mathbb{P}(X = 1) = 3/5,\\ \mathbb{P}(Y = 8) &= \mathbb{P}(X = -2) + \mathbb{P}(X = 2) = 3/20.\end{aligned}$$

You could equally well express this as a table with 3 columns.

Q2. (a) Let $\mathbb{P}(Z = 0) = \mathbb{P}(Z = 3) = a$, $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 2) = b$. Then $2a + 2b = 1$ so $a + b = 1/2$. We have

$$\mathrm{E}(Z) = 0a + b + 2b + 3a = 3a + 3b = 3(a + b) = 3/2.$$

Alternatively, Proposition 11.2(c) tells us that if a random variable is symmetrically distributed about a real number b then its expectation is b. In this case Z is symmetrically distributed about $3/2$ so $\mathrm{E}(Z) = 3/2$.
(b)

$$\mathrm{Var}(Z) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = (0 \times a + 1 \times b + 4 \times b + 9 \times a) - 9/4 = 9a + 5b - 9/4.$$

Since a and b are both probabilities we have $a, b \geq 0$ and $2a+2b = 1$. It is not difficult to see that the smallest $9a + 5b - 9/4$ can be is when $a = 0$ and $b = 1/2$. Then Z has probability mass function $\mathbb{P}(Z = 0) = \mathbb{P}(Z = 3) = 0$, $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 2) = 1/2$. Similarly, for the largest variance, we take $a = 1/2$ and $b = 0$. Then Z has probability mass function $\mathbb{P}(Z = 0) = \mathbb{P}(Z = 3) = 1/2$, $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 2) = 0$.
Note that these agree with the idea that smallest variance means most sharply concentrated while largest variance means most spread out.

(c) We have from part (b) that $\text{Var}(Z) = 9a + 5b - 9/4$. Since $\text{Var}(Z) = 1$, this gives $9a + 5b = 13/4$. We also have $a + b = 1/2$. We may solve these simultaneous equations for a and b to deduce that $a = 3/16$ and $b = 5/16$ giving probability mass function

n	0	1	2	3
$P(Z = n)$	3/16	5/16	5/16	3/16

Q3* (a)
(i) The number of blue marbles I select can be 0, 1 or 2. Thus

$$Range(B) = \{0, 1, 2\}.$$

Let $X = \{b_1, b_2, r_1, r_2, r_3, r_4, r_5, r_6\}$ and let S be the set of all unordered selections of five distinct elements from X. Then $|S| = \binom{8}{5}$. The event that we choose no blues marbles has cardinality $\binom{6}{5}$. Hence $\mathbb{P}(B = 0) = \binom{6}{5}/\binom{8}{5}$. Similarly $\mathbb{P}(B = 1) = \binom{6}{4} \times \binom{2}{1}/\binom{8}{5}$, and $\mathbb{P}(B = 2) = \binom{6}{3}/\binom{8}{5}$. This gives the following probability mass function for B.

n	0	1	2
$P(B = n)$	6/56	30/56	20/56

Hence

$$\text{E}(B) = 0 \times (6/56) + 1 \times (30/56) + 2 \times (20/56) = 5/4.$$

We also have

$$\text{E}(B^2) = 0 \times (6/56) + 1 \times (30/56) + 4 \times (20/56) = 110/56,$$

so $\text{Var}(X) = \text{E}(B^2) - \text{E}(B)^2 = (110/56) - (25/16) = 45/112$.

(ii) We have $R = 5 - B$. Hence Propositions 11.2 and 11.3 give

$$\text{E}(R) = \text{E}(5 - B) = 5 - \text{E}(B) = 5 - 5/4 = 15/4$$

and

$$\text{Var}(R) = \text{Var}(5 - B) = \text{Var}(B) = 45/112.$$

(b) Since $X(s) \geq m$ for all $s \in S$ we have $r \geq m$ for all $r \in Range(X)$. Thus

$$\begin{aligned} \mathrm{E}(X) &= \sum_{r \in Range(X)} r\, \mathbb{P}(X = r) \\ &\geq \sum_{r \in Range(X)} m\, \mathbb{P}(X = r) \\ &= m \sum_{r \in Range(X)} \mathbb{P}(X = r) \\ &= m \end{aligned}$$

where the last equality follows from Proposition 11.1.
The proof that $\mathrm{E}(X) \leq M$ is similar.

AQ1

n	0	1	2
$P(X = n)$	1/8	1/4	5/8

Please let me know if you have any comments or corrections